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OPTIMAL DELAY ESTIMATION IN A MULTIPLE SENSOR ARRAY
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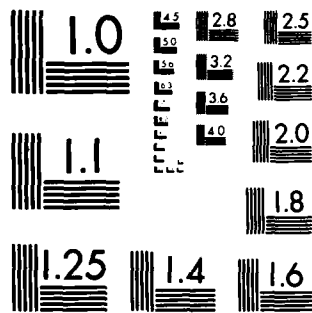
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20. (ABSTRACT Cont.)

discussed. The unknowns are in one case the $M-1$ delays referred to the first sensor and in another case an $M-1$ dimensional subset of independent delays from the $M(M-1)/2$ pairwise delays. The Fisher information matrix (FIM) for the estimates is also derived. The Cramer Rao Matrix Bound (CRMB), which is the inverse of FIM, will show optimal estimator covariances; these are different than the covariances of correlator delay estimators derived by Hahn [4]. Computer evaluations are given for CRMB elements with varied SNR and noise covariance values typical of turbulent boundary layer noise in towed arrays.

Annual Report

Project Title: Investigation of model parameters on delay estimates via sequential state estimation.

Contract No.: N00014-82-K-0048

Principal Investigator: Professor R. Lynn Kirlin

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Laramie, Wyoming 82071
307-766-6137

Long Range Scientific Objectives:

The accuracy of the location parameters is a function of delay measurement variance, apriori variance, and number of sensors. Depending on the other parameters, the number of sensors can be traded against accuracy and cost.

Summary of Work, 1 Nov. 1982 - 30 Sept. 1983:

The second year of this contract has considered the effects of spatially correlated noise on delay estimation in linear arrays. The nonlinear equations for the multiple (M-1) Maximum Likelihood (ML) estimators have been derived. Either delays with respect to the 1st (end) sensor or any independent M-1 delays in the array are usable as unknowns. Cramer-Rao Matrix Bound elements have been formulated, computed, and plotted for a number of realistic values of noise correlation and other parameters.

The variation of the variance bound with correlation is not as significant as the variation with array look angle; up to 5 dB differences were noted.

Most significantly it has been shown that adding more sensors is not always fruitful when spatial noise correlation is present. In the ranges of our parameters little is to be gained in a change from 9 to 15 sensors compared to the change from 3 to 9. More data production would be useful on this subject.

A considerably briefer version of the enclosed report will be submitted to ASSP for publication. Last year's work has been revised per reviewer's suggestions and is re-submitted.

Opportunity was taken following ICASSP-'83 (where a paper was given on the early work of this years research) to visit NUSC in New London in

in April. There discussions with a number of people solidified some of the details and focus of this years work. In particular, the work and data involving spatially correlation noise in towed arrays was brought to attention. This allowed practical values of parameters and clustered configurations to be considered.

ABSTRACT

OPTIMAL DELAY ESTIMATION IN A MULTIPLE
SENSOR ARRAY HAVING SPATIALLY CORRELATED NOISE

by

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The maximal likelihood (ML) estimation of time-of-arrival differences for signals from a single source or target arriving at $M \geq 2$ sensors has been the subject of a large number of papers in recent years. These time differences or delays enable target location. Nearly all previous work has assumed noises which are independent among all sensors. Herein noises are taken to have complex correlation between sensors. A set of nonlinear equations in the unknown delays is derived and possible simplifications discussed. The unknowns are in one case the $M-1$ delays referred to the first sensor and in another case an $M-1$ dimensional subset of independent delays from the $M(M-1)/2$ pairwise delays. The Fisher information matrix (FIM) for the estimates is also derived. The Cramer Rao Matrix Bound (CRMB), which is the inverse of FIM, will show optimal estimator covariances; these are different than the covariances of correlator delay estimators derived by Hahn [4]. Computer evaluations are given for CRMB elements with varied SNR and noise covariance values typical of turbulent boundary layer noise in towed arrays.

September 30, 1983



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OPTIMAL DELAY ESTIMATION IN A MULTIPLE SENSOR ARRAY HAVING SPATIALLY CORRELATED NOISE⁺

I. Introduction

The estimation of time-of-arrival differences for signals from a single source or target arriving at multiple sensors has been the subject of a considerable number of papers in recent years. These time delay differences, or simply delays, enable target localization through straightforward geometrical considerations when the signal path is non-dispersive [1,2]. Although target location is the primary goal, delay estimation is essentially equivalent as there is a one-one*, although nonlinear, relation between the maximum-likelihood (ML) delay vector and ML location vector.

Essentially all of the results of available literature (except [9] have been based not only upon the geometric and non-dispersive assumptions stated above but also upon noise spectra which are independent among sensors. The independent noise assumption is adequate if either the sensor self-noise is dominant or the sensors are spatially separated sufficiently such that the environmental noise is indeed independent or uncorrelated among sensors. However, this is not always a reasonable assumption and the effects of spatially correlated noise in the estimation of delays and delay variances must be considered. Thus appropriate analyses are herein undertaken to consider correlated noise from diffuse sources. Results are compared to those previously published for independent sensor noises.

Owsley and Fay [11] have considered correlated noise when clustering sensors and optimizing beamformers. The comparable optimization of delay estimation has not previously been approached. By choosing the correlation parameter ρ , we may include the proportionality of correlated turbulent boundary layer tow-noise and isotropic sensor noise.

The basic approach is to assume that complex Fourier coefficients $X_i(k)$ at the i^{th} sensor for the k^{th} frequency are available, having been obtained from T-second time records, where T is long with respect to the signal correlation time.

* For an array with three sensors in line there is an ambiguity in the sign of bearing angle, which we assume may be solved with additional information.

+ This study funded under office of Naval Research, contract number N00014-82-K-0048.

The time data records are

$$x_i(t) = s(t-d_i) + n_i(t), \quad i = 1, 2, \dots, M. \quad (1)$$

where d_i are the delays from the reference sensor to the i^{th} sensor ($d_1 = 0$), $s(t)$ is a zero-mean, Gaussian, stationary signal, and $n_i(t)$ is the additive Gaussian noise at the i^{th} sensor.

II. Background

The problem of delay vector estimation for multiple sensors has been studied with the above approach in original papers by Hahn and Tretter [3], Hahn [4] and Schultheiss [5]. Closely following their presentations, let

$$X_i(k) = \frac{1}{T} \int_{-T/2}^{T/2} x_i(t) \exp\{-jk\omega_0 t\} dt, \quad k = 1, 2, \dots, K, \quad (2)$$

where $\omega_0 = 2\pi/T$. Define a vector X containing the above MK Fourier coefficients as elements. If $S(\omega)$ and $N_i(\omega)$ are the signal and noise spectra at the i^{th} sensor, the probability density for x can be written

$$p(\underline{X}) = \left(\pi^{MK} \prod_{k=1}^K \det R(k) \right)^{-1} \exp\left[-\sum_{k=1}^K \underline{X}^T(k) R^{-1}(k) \underline{X}(k)\right] \quad (3)$$

where

$$\underline{X}(k) = [X_1(k), X_2(k), \dots, X_M(k)]^T$$

$$\underline{X} = [\underline{X}^T(1); \underline{X}^T(2), \dots, \underline{X}^T(K)]^T$$

$$\underline{V}(k) = [1, \exp(-jk\omega_0 d_2), \dots, \exp(-jk\omega_0 d_M)]^T$$

$$N(k) = [N_{ij}(k)], \text{ an } M \times M \text{ matrix of noise}$$

cross-power spectra

$$R(k) = N(k) + S(k) \underline{V}^*(k) \underline{V}^T(k)$$

and where $*$ superscript denotes complex conjugation.

In order to obtain the ML estimate of delays, determinant and inverse theorems of use are

$$|R| = |N + S \underline{V} \underline{V}^T| = |N| |I + N^{-1} S \underline{V} \underline{V}^T| = |N| (1 + \underline{V}^T N^{-1} S \underline{V}) \quad (4)$$

and

$$R^{-1} = N^{-1} - N^{-1} V^* (V^T N^{-1} V^* + 1/S)^{-1} V^T N^{-1} \quad (5)$$

Defining elements of N^{-1} as N^{ik} , the likelihood function of the delay vector $D^T = (d_2, d_3, \dots, d_M)$ is, using (4),

$$\begin{aligned} \Lambda = \ln p(\underline{X}) = -\ln(\pi^{MK}) - \sum_{B+} \ln[|N| (1 + SV^T N^{-1} V^*)] \\ - \sum_{B+} [X^T N^{-1} X^* - \frac{X^T N^{-1} V^* V^T N^{-1} X^*}{V^T N^{-1} V^* + 1/S}] \end{aligned} \quad (6)$$

where \sum_{B+} means sum over positive frequencies.

Hahn and Tretter [3] have shown that, when N is diagonal, $[N_1, N_2, \dots, N_M]$, the Fisher information matrix for D (FIM = $-\langle \text{grad} \langle \text{grad} \ln \Lambda \rangle \rangle$ where $\langle \cdot \rangle$ is expected value) is

$$\text{FIM} = \sum_{B+} 2\omega^2 \frac{S^2}{1 + \sum S/N_i} [(\text{tr } N^{-1})_{N_p}^{-1} - N_p^{-1} 11^T N_p^{-1}] \quad (7)$$

where N_p^{-1} is N^{-1} with the first row and column removed. The Cramer-Rao Matrix bound (CRMB) for the delays D is $(\text{FIM})^{-1}$. The ML estimate covariance is known to asymptotically approach the CRMB. The ML estimate for small delays (\hat{D} is the error when $D = 0$) and independent noise is

$$\hat{D} = -\langle C \rangle^{-1} B^T, \quad (8)$$

where

$$\langle C \rangle^{-1} = \text{FIM}, \quad (9)$$

$$B = \sum_{B+} j\omega \frac{S}{1 + \sum S/N_i} 1^T N^{-1} [X X_p^* - X^* X_p^T] N_p^{-1}, \quad (10)$$

and x_p is $X(k)$ with the first element ($X_1(k)$) removed.

Hahn and Tretter also show that the ML D estimate can be implemented either as a beamformer (ideally in real time only when the N_i are proportional, because of phase-matching filter criteria), or as a cross correlator

system which produces the $M(M-1)/2$ delay estimates. The correlator system has cross-spectral filters

$$|F_{ij}|^2 = \frac{S/N_i N_j}{1 + \sum_k S/N_k} \quad , \quad i, j = 1, 2, \dots, M, \quad i \neq j. \quad (11)$$

The error covariance matrix for the pair-wise delay estimates of the correlators is shown by Hahn [4] to have elements

$$\text{var}(\hat{d}_{ij}) = \frac{2\pi}{T} \frac{\int_B \omega^2 |F_{ij}|^4 [N_i N_j + S(N_i + N_j)] d\omega}{(\int_B \omega^2 |F_{ij}|^2 S d\omega)^2} \quad (12)$$

$$\text{covar}(\hat{d}_{ij}, \hat{d}_{k\ell}) = 0, \quad i, j, k, \ell \text{ all distinct}$$

$$\begin{aligned} \text{covar}(\hat{d}_{ij}, \hat{d}_{i\ell}) &= \frac{2\pi}{T} \frac{\int_B \omega^2 |F_{ij}|^2 |F_{i\ell}|^2 S N_i d\omega}{\int_B \omega^2 |F_{ij}|^2 S d\omega \int_B \omega^2 |F_{i\ell}|^2 S d\omega}, \quad i \neq \ell \\ &= -\text{covar}(\hat{d}_{ij}, \hat{d}_{\ell i}), \quad j \neq \ell \end{aligned} \quad (13)$$

It is emphasized that these are correlator error covariances of the \hat{d}_{ij} and not ML estimator error variances, which are derived herein.

The delays having covariance matrix defined by (12) and (13) are not the $M-1$ delays referred to a single sensor. Hahn and Tretter have shown how to use weighted linear combinations of the $M(M-1)/2$ cross correlation delay estimates, \hat{d}_{ij} , to form an estimate for $D = (d_i)$ which achieves the CRMB of (7).

With independent noises maximization of Λ in (6) over the vector D concentrates on the second term in the second summation, because other terms are not dependent on the d_i . This is not generally the case, and an analytical solution is not available, as was pointed out in the multipath analysis given by Owsley [6]. However, the generation of a set of nonlinear equations in the unknowns d_{ij} may be obtained.

In the next section ML estimator equations for the $M(M-1)/2$ pairwise delays are derived. Section IV produces the CRMB for these delays. Section V considers the $M-1$ delays $d_i - d_1$, and Section VI derives the CRMB for the $M-1$ delays $d_i - d_1$.

III. Estimation of an Independent Subset of $M(M-1)/2$ Delays

This section will determine equations for ML estimates of an independent subset of the $M(M-1)/2$ delays $d_i - d_k = d_{ik}$. In contrast most other papers referenced find ML estimates of either the $M-1$ delays $(d_i - d_1)$, $2 \leq i \leq M$, or other parameters such as range and bearing, functions of which the d_i may be written. The reason for our choosing the d_{ik} is that a-priori information about linear relationships among them may subsequently be used as in [10] to improve the delay estimates $\hat{d}_i - d_1$ or any other subset.

Because we will find equations for real variables and real unknowns, and we wish to be able to show effects of correlated noise on various parts of the formulations, double sums throughout the paper are usually broken into several pieces.

Now consider the two summations in (6), the only functions of D ,

$$\Lambda' = -\sum_{B+} \ln[(1 + S \sum_p N^{pp} + S \sum_p \sum_{q \neq p} N^{pq} e^{-j\omega(d_p - d_q)}) |N|] \\ \sum_{B+} \sum_i \sum_k X_i [N^{ik} - \sum_t \sum_r \frac{N^{ir} N^{tk} e^{j\omega(d_r - d_t)}}{\sum_p N^{pp} + \sum_p \sum_{q \neq p} N^{pq} e^{j\omega(d_p - d_q)} + \frac{1}{S}}] X_k^* \quad (14)$$

Thus we would like to solve for the \hat{d}_{ik} which maximizes

$$\Lambda'' = -\sum_{B+} \ln(1/S + \sum_p \sum_q N^{pq} e^{-j\omega d_{pq}}) \\ + \sum_{B+} \sum_m \sum_n X_m X_n^* \sum_r \sum_t \frac{N^{mr} N^{tn} e^{j\omega d_{rt}}}{1/S + \sum_p \sum_q N^{pq} e^{-j\omega d_{pq}}} \quad (15)$$

$$= \sum_{B+} [-\ln(g) + \frac{X^T N^{-1} V^* V^T N^{-1} X^*}{g}] \quad (16)$$

where, using (15),

$$g = 1/S + V^T N^{-1} V^* = 1/S + \sum_p N^{pp} + 2 \sum_p \sum_{q > p} (\cos \omega d_{pq} \operatorname{Re}\{N^{pq}\} \\ + \sin \omega d_{pq} \operatorname{Im}\{N^{pq}\}) \quad (17)$$

Differentiating Λ'' with respect to d_{ik} (assuming all d_{ik} independent) gives

$$\frac{\partial \Lambda''}{\partial d_{ik}} = \sum_{B+} \left[\frac{\partial(V^T N^{-1} V^*)}{\partial d_{ik}} \right. \\ \left. \frac{g X^T N^{-1} \frac{\partial(V^* V^T) N^{-1} X^*}{\partial d_{ik}} - X^T N^{-1} V^* V^T N^{-1} X^* \frac{\partial(V^T N^{-1} V^*)}{\partial d_{ik}}}{g^2} \right] \quad (18)$$

Setting this equal to zero and rearranging gives

$$\sum_{B+} \left\{ -\frac{\partial(V^T N^{-1} V^*)}{\partial d_{ik}} \cdot \frac{1}{g} + X^T N^{-1} \left[\frac{\partial(V^* V^T)}{\partial d_{ik}} \cdot \frac{1}{g} - \frac{V^* V^T}{g^2} \cdot \frac{\partial(V^T N^{-1} V^*)}{\partial d_{ik}} \right] N^{-1} X^* \right\} = 0 \quad (19)$$

The square-bracketed terms in (19) are a matrix $U(i,k) = (u_{mn}(i,k))$. A typical element u_{mn} has values which differ according to whether or not $(m,n) = (i,k)$ or (k,i) . Using

$$(V^* V^T)_{m,n} = e^{j\omega d_{mn}}, \quad (20a)$$

$$\frac{\partial(V^* V^T)}{\partial d_{ik}}_{m,n} = \begin{cases} j\omega e^{j\omega d_{ik}}, & (m,n) = (i,k) \\ -j\omega e^{-j\omega d_{ik}}, & (m,n) = (k,i) \\ 0, & (m,n) \neq (i,k), (k,i) \end{cases} \quad (20b)$$

and

$$\begin{aligned} g_1 &= \frac{\partial V^T N^{-1} V^*}{\partial d_{ik}} = j\omega(-N^{ik} e^{-j\omega d_{ik}} + N^{ki} e^{j\omega d_{ik}}) \\ &= 2\omega(-\text{Re}\{N^{ik}\} \sin \omega d_{ik} + \text{Im}\{N^{ik}\} \cos \omega d_{ik}) \end{aligned} \quad (20c)$$

in $u_{mn}(i,k)$ give

$$u_{mn} = \frac{1}{g} \begin{cases} e^{j\omega d_{ik}} (j\omega - g_1/g) & , (m,n) = (i,k) \\ e^{-j\omega d_{ik}} (-j\omega - g_1/g) & , (m,n) = (k,i) \\ -e^{j\omega d_{mn}} g_1/g & , (m,n) \neq (i,k), (k,i) \end{cases} \quad (21)$$

Insertion of (21) into (19) constitutes $M(M-1)/2$ equations, nonlinear, to be solved for the d_{ik} . This is pursued further in Appendix A for diffuse noise. Note that only $M-1$ delays can be independent. We now turn our attention to the CRMB.

IV. The Cramer-Rao Bound for an Independent Subset of the $M(M-1)/2$ Delay Estimates of d_{ik}

As is well known, maximum likelihood estimators have variances which approach the Cramer-Rao bound. The variance bounds for the \hat{d}_{ik} are the elements in the diagonal of

$$\text{CRMB} = (\text{FIM})^{-1} = (-\langle \text{grad}(\text{grad } \ln \Lambda')^T \rangle)^{-1}, \quad (22)$$

wherein FIM is the Fisher Information Matrix, $\text{grad } \Lambda'$ is a row vector whose m^{th} element is the derivative of Λ' with respect to the m^{th} delay (the m^{th} d_{ik} here), Λ' is the expression in Eq. (17), and $\langle \cdot \rangle$ denotes expectation. The outer gradient operator creates a matrix whose elements are $\frac{\partial}{\partial d_{rt}} \left(\frac{\partial \Lambda'}{\partial d_{ik}} \right)$. We have already found the inner partial -- the result is Eq. (19) and following. For any $M-1$ independent delays the following applies.

$$\begin{aligned} & \text{Taking and negating the second partial with respect to } d_{rt} \text{ gives} \\ & - \frac{\partial}{\partial d_{rt}} \left(\frac{\partial \Lambda'}{\partial d_{ik}} \right) = - \sum_{B+} \left\{ - \left[g \frac{\partial}{\partial d_{rt}} \left(\frac{\partial V_N^{T-1} V^*}{\partial d_{ik}} \right) - \frac{\partial (V_N^{T-1} V^*)}{\partial d_{ik}} \frac{\partial (V_N^{T-1} V^*)}{\partial d_{rt}} \right] \right. \\ & \quad \left. + \frac{1}{g^2} \left[g X_N^{T-1} \frac{\partial}{\partial d_{rt}} \left(\frac{\partial V^* V^T}{\partial d_{ik}} \right) N^{-1} X^* - X_N^{T-1} \frac{\partial (V^* V^T)}{\partial d_{ik}} N^{-1} X^* \frac{\partial (V_N^{T-1} V^*)}{\partial d_{rt}} \right. \right. \\ & \quad \left. \left. - X_N^{T-1} V^* V^T N^{-1} X^* \frac{\partial}{\partial d_{rt}} \left(\frac{\partial V_N^{T-1} V^*}{\partial d_{ik}} \right) \right. \right. \\ & \quad \left. \left. - \frac{\partial (V_N^{T-1} V^*)}{\partial d_{ik}} X_N^{T-1} \frac{\partial (V^* V^T)}{\partial d_{rt}} N^{-1} X^* \right] \right. \\ & \quad \left. + \frac{1}{g^4} \left[(X_N^{T-1} V^* V^T N^{-1} X^* \frac{\partial (V_N^{T-1} V^*)}{\partial d_{ik}}) + 2g \frac{\partial (V_N^{T-1} V^*)}{\partial d_{rt}} \right] \right\} \\ & = - \sum_{B+} \frac{1}{g} \left\{ - \left[\frac{\partial g_1}{\partial d_{rt}} - \frac{g_1 g_2}{g} \right] + X_N^{T-1} \left[\frac{\partial^2 (V^* V^T)}{\partial d_{rt} \partial d_{ik}} - \frac{g_2}{g} \frac{\partial (V^* V^T)}{\partial d_{ik}} \right. \right. \\ & \quad \left. \left. - \frac{1}{g} \frac{\partial g_1}{\partial d_{rt}} V^* V^T - \frac{1}{g} g_1 \frac{\partial (V^* V^T)}{\partial d_{rt}} + \frac{2}{g} g_1 g_2 V^* V^T \right] N^{-1} X^* \right\} \\ & = \sum_{B+} \frac{1}{g} (g_{12} - g_1 g_2 / g) - \frac{1}{g} \sum_{B+} X_N^{T-1} B_N^{-1} X^* \end{aligned} \quad (23)$$

where

$$g_1 = \frac{\partial g}{\partial d_{ik}} = 2\omega(-\operatorname{Re}\{N^{ik}\}\sin\omega d_{ik} + \operatorname{Im}\{N^{ik}\}\cos\omega d_{ik}) \quad (24)$$

$$g_2 = \frac{\partial g}{\partial d_{rt}} = 2\omega(-\operatorname{Re}\{N^{rt}\}\sin\omega d_{rt} + \operatorname{Im}\{N^{rt}\}\cos\omega d_{rt}) \quad (25)$$

and $B = (b_{mn})$ is found using

$$g_{12} = \frac{\partial^2 g}{\partial d_{rt} \partial d_{ik}} = \begin{cases} 0, (r,t) \neq (i,k) \\ 2\omega^2(-\operatorname{Re}\{N^{ik}\}\cos\omega d_{ik} - \operatorname{Im}\{N^{ik}\}\sin\omega d_{ik}), (r,t)=(i,k) \end{cases} \quad (26)$$

$$\left(\frac{\partial^2 (V \cdot V^T)}{\partial d_{rt} \partial d_{ik}} \right)_{mn} = \begin{cases} 0, (r,t) \neq (i,k) \\ -\omega^2 e^{j\omega d_{ik}}, (m,n) = (i,k) \\ -\omega^2 e^{-j\omega d_{ik}}, (m,n) = (k,i) \\ 0, (m,n) \neq (k,i), (i,k) \end{cases}, (r,t)=(i,k) \quad (27)$$

Then b_{mn} has the following values:

$$\underline{(m,n) = (i,k), (k,i)} \\ b_{ik} = \left(\frac{-g_2}{g} j\omega - \mu(\omega^2 + \frac{g_{12}}{g}) + \frac{g_1}{g} j\omega \right) + 2 \frac{g_1 g_2}{g^2} e^{j\omega d_{ik}} \quad (28a)$$

$$b_{ki} = b_{ik}^*$$

where $\mu = 0$ if $(r,t) \neq (i,k)$, $\mu = 1$ if $(r,t) = (i,k)$

$$\underline{(m,n) = (r,t), (t,r); (r,t) \neq (i,k)}$$

$$b_{rt} = \left(\frac{-g_1}{g} j\omega + \frac{2g_1 g_2}{g^2} \right) \quad (28b)$$

$$b_{tr} = b_{rt}^*$$

$$\underline{(m,n) \neq (i,k), (k,i), (r,t), (t,r)}$$

$$b_{mn} = \left(\frac{2g_1 g_2}{g^2} - \mu \frac{g_{12}}{g} \right) e^{j\omega d_{mn}} \quad (28c)$$

$$b_{nm} = b_{mn}^*$$

Using

$$\langle X_r X_t^* \rangle = \begin{cases} Se^{-j\omega d} r_t + N_{rt}^* & , r \neq t \\ S + N_1 & , r = t \end{cases} \quad (29)$$

and writing

$$\begin{aligned} X^T N^{-1} B N^{-1} X^* &= \sum_m b_{mm} [|X_m|^2 (N^{mm})^2 \\ &+ 2 \operatorname{Re}\{X_m N^{mm} \sum_{q \neq m} X_q^* N^{mq}\} \\ &+ \sum_{p \neq m} \sum_{q \neq m} X_p X_q^* N^{pm} N^{mq}] \\ &+ \sum_m \sum_{n > m} [\operatorname{Re}\{b_{mn}\} \operatorname{Re}\{G(m,n)\} \\ &- \operatorname{Im}\{b_{mn}\} \operatorname{Im}\{G(m,n)\}] \end{aligned} \quad (30)$$

where

$$G(m,n) = \sum_p X_p N^{pm} \sum_q X_q^* N^{qn} \quad (31)$$

The elements of the FIM are

$$\begin{aligned} (FIM)_{rt,ik} &= \sum_{B+} \frac{1}{g} (g_{12} - g_1 g_2 / g) \\ &- \sum_{B+} \frac{1}{g} \{ \sum_m b_{mm} [(S+N_1) (N^{mm})^2 \\ &+ 2 \sum_{q \neq m} ((S \cos \omega d_{mq} + \operatorname{Re}\{N_{mq}^*\}) \operatorname{Re}\{N^{mm} N^{mq}\} \\ &- (-S \sin \omega d_{mq} + \operatorname{Im}\{N_{mq}^*\}) \operatorname{Im}\{N^{mm} N^{mq}\}) \\ &+ 2 \sum_{p \neq m} \sum_{q > p} ((S \cos \omega d_{pq} + \operatorname{Re}\{N_{pq}^*\}) \operatorname{Re}\{N^{pm} N^{mq}\} \\ &- (-S \sin \omega d_{pq} + \operatorname{Im}\{N_{pq}^*\}) \operatorname{Im}\{N^{pm} N^{mq}\}) \\ &+ \sum_{p \neq m} |N^{pm}|^2 (S \cos \omega d_{pm} + \operatorname{Re}\{N_{pm}^*\}) \} \\ &+ 2 \sum_m \sum_{n > m} [\operatorname{Re}\{b_{mn}\} < \operatorname{Re}\{G(m,n)\} > \\ &- \operatorname{Im}\{b_{mn}\} < \operatorname{Im}\{G(m,n)\} >] \end{aligned} \quad (32)$$

where

$$\begin{aligned} \langle \text{Re}\{G(m,n)\} \rangle &= \sum_p \sum_q (\text{Re}\{N^{pm} N^{nq}\} (S \cos \omega d_{pq} + \text{Re}\{N_{pq}^*\}) \\ &\quad - \text{Im}\{N^{pm} N^{nq}\} (-S \sin \omega d_{pq} + \text{Im}\{N_{pq}^*\})) \end{aligned} \quad (33)$$

and

$$\begin{aligned} \langle \text{Im}\{G(m,n)\} \rangle &= \sum_p \sum_q (\text{Re}\{N^{pm} N^{nq}\} (-S \sin \omega d_{pq} + \text{Im}\{N_{pq}^*\}) \\ &\quad + \text{Im}\{N^{pm} N^{nq}\} (S \cos \omega d_{pq} + \text{Re}\{N_{pq}^*\})) \end{aligned}$$

Use of these elements in the FIM is restricted for inversion to the CRMB to M-1 independent delays. Further examination of (32) for diffuse noise is given in Appendix C.

We now investigate for the $M-1$ d_i changes in these earlier results caused by consideration of noise which is spatially correlated.

V. Maximum Likelihood Estimation of the $M-1$ Delays $d_i - d_1$

We again maximize Λ by maximizing

$$\Lambda' = - \sum_{B+} \ln g + \sum_{B+} X_N^T N^{-1} V^* V^T N^{-1} X^* / g \quad (34)$$

Writing

$$X_N^T N^{-1} = (\sum_p X_p N^{p1}, \sum_p X_p N^{p2}, \dots, \sum_p X_p N^{pm}) \quad (35)$$

and the m, n th element

$$(V^* V^T)_{m,n} = v_{mn} = e^{-j\omega(d_m - d_n)}$$

gives

$$\begin{aligned} X_N^T N^{-1} V^* V^T N^{-1} X^* / g &= \frac{1}{g} \sum_m \sum_n v_{mn} (\sum_p X_p N^{pm} \sum_q X_q^* N^{qn}) \\ &= \frac{1}{g} \sum_m \sum_n v_{mn} (X_m N^{mm} + \sum_{p \neq m} X_p N^{pm}) (X_n^* N^{nn} + \sum_{q \neq n} X_q^* N^{qn}) \\ &= \frac{1}{g} \sum_m \sum_n v_{mn} (X_m X_n^* N^{mm} N^{nn} + X_m N^{mm} \sum_{q \neq n} X_q^* N^{qn} \\ &\quad + X_n^* N^{nn} \sum_{p \neq m} X_p N^{pm} \\ &\quad + \sum_{p \neq m} \sum_{q \neq n} X_p X_q^* N^{pm} N^{qn}). \end{aligned} \quad (36)$$

In this form it may be seen that (34) differs from the spatially uncorrelated noise case only in the $-\sum_{B+} \ln g$ term and the terms in parentheses in (36) other than $X_m X_n^* N^{mm} N^{nn}$. If $\rho = 0$, $\ln g$ is not a function of the delays and all N^{pq} , $p \neq q$, are zero. Then as the literature cited shows [3,4], maximization of (6) reduces to either a beamformer (choosing $M-1$ d_i) or a system of $M(M-1)/2$ correlators (choosing $d_i - d_k$).

Further manipulation of Eq. (36) when noise is diffuse is given in App. A for a special "worst case" when all correlations are real and equal.

Maximizing Λ means solving for d_i in

$$\frac{\partial \Lambda}{\partial d_i} = - \sum_{B+} \frac{1}{g} \frac{\partial g}{\partial d_i} + \sum_{B+} \left\{ \frac{1}{g} X^T N^{-1} \frac{\partial (V^* V^T)}{\partial d_i} N^{-1} X^* - \frac{X^T N^{-1} V^* V^T N^{-1} X^*}{g^2} \frac{\partial g}{\partial d_i} \right\} = 0. \quad (37)$$

Using

$$g_i = \frac{\partial g}{\partial d_i} = 2\omega \sum_{p \neq i} (-\operatorname{Re}\{N^{ip}\} \sin \omega(d_i - d_p) + \operatorname{Im}\{N^{ip}\} \cos \omega(d_i - d_p)) \quad (38)$$

and

$$\frac{\partial (V^* V^T)}{\partial d_i} = j\omega \begin{pmatrix} & & & & -e^{j\omega(d_1 - d_i)} \\ & & & & -e^{j\omega(d_2 - d_i)} & -0- \\ & & & & \vdots & \\ & & & & \vdots & \\ e^{j\omega(d_i - d_1)} & e^{j\omega(d_i - d_2)} & \dots & 0 & \dots & e^{j\omega(d_i - d_M)} \\ & & & & \vdots & \\ & & & & \vdots & \\ & & & & -e^{j\omega(d_M - d_i)} & -0- \end{pmatrix} \quad (39)$$

in (37) gives

$$\frac{\partial \Lambda}{\partial d_i} = - \sum_{B+} \frac{2\omega}{g} \sum_{p \neq i} (\operatorname{Re}\{N^{ip}\} \sin \omega(d_i - d_p) + \operatorname{Im}\{N^{ip}\} \cos \omega(d_i - d_p)) + \sum_{B+} X^T N^{-1} A N^{-1} X^* \quad (40)$$

where $A = (a_{mn}(i))$ and

$$a_{mn}(i) = \frac{1}{g} \begin{cases} -g_i e^{j\omega(d_m - d_n)/g}; m, n \neq i \text{ or } m = n = i \\ (j\omega - g_i/g) e^{j\omega(d_i - d_n)}; m = i, n \neq i \\ (-j\omega - g_i/g) e^{-j\omega(d_i - d_m)}; m \neq i, n = i \end{cases} \quad (41)$$

Now note that

$$\begin{aligned}
 X^T N^{-1} A N^{-1} X^* &= \sum_m \sum_n a_{mn}(i) (X_m X_n^* N^{mm} N^{nn} \\
 &+ X_m N^{mm} \sum_{q \neq n} X_q^* N^{qn} + X_n^* N^{nn} \sum_{p \neq m} X_p N^{pm} \\
 &+ \sum_{p \neq m} \sum_{q \neq n} X_p X_q^* N^{pm} N^{qn})
 \end{aligned} \quad (42)$$

Ignoring the first summation in (40) (from the $\ln g$ term of Λ) and setting (42) equal to zero for $i = 2, 3, \dots, M$ constitutes $M-1$ nonlinear equations in the $M-1$ unknowns d_2, d_3, \dots, d_m ($d_1 = 0$).

Equations (42) may be made real by observing that every (m, n) term has its conjugate. Thus

$$\begin{aligned}
 X^T N^{-1} A N^{-1} X^* &= \sum_m a_{mm}(i) (|X_m|^2 (N^{mm})^2 + 2 \operatorname{Re}\{X_m N^{mm} \sum_{q \neq m} X_q^* N^{mq}\} \\
 &+ \sum_{p \neq m} \sum_{q \neq m} X_p X_q^* N^{pm} N^{mq}) \\
 &+ 2 \sum_m \sum_{n > m} (\operatorname{Re}\{a_{mn}(i)\} \operatorname{Re}\{G(m, n)\} \\
 &- \operatorname{Im}\{a_{mn}(i)\} \operatorname{Im}\{G(m, n)\})
 \end{aligned} \quad (43)$$

where

$$G(m, n) = \sum_p X_p N^{pm} \sum_q X_q^* N^{nq}. \quad (44)$$

Because the $a_{mn}(i)$ are functions of g_i and g , and g_i and g are functions of all delay differences $d_p - d_q$, the solution for d_i cannot be found in terms of X_i and X_1 alone nor even as a linear combination of the $X_p X_q^* e^{j\omega(d_p - d_q)}$ correlators.

VI. The Cramer-Rao Matrix Bound for the $M-1$ Delays $d_i - d_1$.

It is well known that ML estimators approach the Cramer-Rao bound (CRMB). The variance bounds for the delay estimates \hat{d}_i are the diagonal elements in

$$\text{CRMB} = (\text{FIM})^{-1} = (-\langle \text{grad} (\text{grad } \Lambda')^T \rangle)^{-1}, \quad (45)$$

wherein FIM is the Fisher Information Matrix, $\text{grad } \Lambda'$ is a row vector whose m^{th} element is the derivative of Λ' with respect to the m^{th} delay d_{m+1} , Λ' is the expression in (16), and $\langle \cdot \rangle$ denotes expectation. The outer gradient operator creates a matrix whose elements are $-\frac{\partial}{\partial d_k} \left(\frac{\partial \Lambda}{\partial d_i} \right)$. The inner partial has already been given by (37) and (40). Continuing we find with $\frac{\partial}{\partial d_k} = g_k$ that

$$\begin{aligned}
 -\frac{\partial}{\partial d_k} \left(\frac{\partial \Lambda}{\partial d_i} \right) &= - \sum_{B+} \frac{1}{g} \left\{ - \left[\frac{\partial g_i}{\partial d_k} - \frac{g_i g_k}{g} \right] \right. \\
 &\quad + X^T N^{-1} \left[\frac{\partial^2 (V^* V^T)}{\partial d_k \partial d_i} - \frac{g_k}{g} \frac{\partial (V^* V^T)}{\partial d_i} \right. \\
 &\quad \left. - \frac{1}{g} \frac{\partial g_i}{\partial d_k} V^* V^T - \frac{1}{g} g_i \frac{\partial (V^* V^T)}{\partial d_k} \right. \\
 &\quad \left. \left. + \frac{2}{g} g_i g_k V^* V^T \right] N^{-1} X^* \right\} \quad (46)
 \end{aligned}$$

If $k \neq i$, then

$$g_{ik} = \frac{\partial g_i}{\partial d_k} = 2\omega^2 (\text{Re}\{N^{ik}\} \cos \omega(d_i - d_k) + \text{Im}\{N^{ik}\} \sin \omega(d_i - d_k)) \quad (47a)$$

and

$$\left(\frac{\partial^2 (V^* V^T)}{\partial d_k \partial d_i} \right)_{m,n} = \begin{cases} 0, & (m,n) \neq (k,i) \text{ or } (i,k) \\ \omega^2 e^{j\omega(d_k - d_i)}, & (m,n) = (k,i) \\ \omega^2 e^{-j\omega(d_k - d_i)}, & (m,n) = (i,k) \end{cases} \quad (47b)$$

These give

$$\begin{aligned}
 -\frac{\partial}{\partial d_k} \left(\frac{\partial \Lambda'}{\partial d_i} \right) &= \sum_{B+} \frac{1}{g} [g_{ik} - g_i g_k / g] \\
 &\quad - \sum_{B+} \frac{1}{g} X^T N^{-1} U N^{-1} X^* \quad (48)
 \end{aligned}$$

where u_{mn} have the following values

$m \neq i, k; n \neq i, k$

$$u_{mn} = \left(-\frac{g_{ik}}{g} + \frac{2g_i g_k}{g} \right) e^{j\omega(d_m - d_n)} \quad (49a)$$

$$(m,n) = (k,i) \text{ or } (i,k)$$

$$u_{ki} = (\omega^2 - \frac{g_{ik}}{g} + \frac{2g_i g_k}{g^2} + j\omega(\frac{g_k}{g} - \frac{g_i}{g})) e^{j\omega(d_k - d_i)} \quad (49b)$$

$$u_{ik} = u_{ki}^*$$

$$m = k; n \neq i, k \text{ or } n = k; m \neq i, k$$

$$u_{kn} = (-\frac{g_{ik}}{g} + \frac{2g_i g_k}{g^2} - \frac{g_i}{g} j\omega) e^{j\omega(d_k - d_n)} \quad (49c)$$

$$u_{mk} = u_{kn}^*$$

$$m = i; n \neq i, k \text{ or } n = i; m \neq i, k$$

$$u_{in} = (-\frac{g_{ik}}{g} + \frac{2g_i g_k}{g^2} - \frac{g_k}{g} j\omega) e^{j\omega(d_i - d_n)} \quad (49d)$$

$$u_{mi} = u_{in}^*$$

$$m = n$$

$$u_{mm} = (\frac{-g_{ik}}{g} + \frac{2g_i g_k}{g^2}) \quad (49e)$$

Now the (FIM) may be written ($i \neq k$)

$$\begin{aligned} (FIM)_{ik} &= \sum_{B+} \frac{1}{g} [g_{ik} - g_i g_k / g] \\ &- \frac{1}{g} < \sum_{B+} [\sum_m u_{mm} (|X_m|^2 (N^{mm})^2 \\ &+ 2 \operatorname{Re}\{X_m N^{mm} \sum_{q \neq m} X_q^* N^{mq}\} \\ &+ \sum_{p \neq m} \sum_{q \neq m} X_p X_q^* N^{pm} N^{mq}) \\ &+ 2 \sum_m \sum_{n > m} (\operatorname{Re}\{u_{mn}\} \operatorname{Re}\{G(m,n)\} \\ &- \operatorname{Im}\{u_{mn}\} \operatorname{Im}\{G(m,n)\})] > \end{aligned} \quad (50)$$

where $G(m,n) = \sum_p X_p N^{pm} \sum_q X_q^* N^{nq}$ as in (44).

Writing all $X_p X_q^*$ as in (29) gives

$$\langle X_p X_q^* \rangle = \begin{cases} S e^{-j\omega(d_p - d_q)} + N_{pq}^*, & p \neq q \\ S + N_1, & p = q \end{cases} \quad (51)$$

$$\langle \text{Re}\{G(m,n)\} \rangle = \sum_p \sum_q (\text{Re}\{N^{pm} N^{nq}\} (S \cos \omega(d_p - d_q) + \text{Re}\{N_{pq}^*\}) - \text{Im}\{N^{pm} N^{nq}\} (-S \sin \omega(d_p - d_q) + \text{Im}\{N_{pq}^*\})) \quad (52)$$

$$\langle \text{Im}\{G(m,n)\} \rangle = \sum_p \sum_q (\text{Re}\{N^{pm} N^{nq}\} (-S \sin \omega(d_p - d_q) + \text{Im}\{N_{pq}^*\}) + \text{Im}\{N^{pm} N^{nq}\} (S \cos \omega(d_p - d_q) + \text{Re}\{N_{pq}^*\})) \quad (53)$$

and

$$\begin{aligned} & \sum_{p \neq m} \sum_{q \neq m} N^{pm} N^{mq} \langle X_p X_q^* \rangle \\ &= 2 \sum_{p \neq m} \sum_{q > p, \neq m} [\text{Re}\{N^{pm} N^{mq}\} (S \cos \omega(d_p - d_q) + \text{Re}\{N_{pq}^*\}) - \text{Im}\{N^{pm} N^{mq}\} (-S \sin \omega(d_p - d_q) + \text{Im}\{N_{pq}^*\})] \\ & \quad + \sum_{p \neq m} (|N^{pm}|^2 (S + N_1)) \end{aligned} \quad (54)$$

giving

$$\begin{aligned} (\text{FIM})_{ik} &= \sum_{B+} \frac{1}{g} (g_{ik} - g_i g_k / g) \\ & - \frac{1}{g} \sum_{B+} \left\{ \sum_m u_{mm} [(S + N_1) (N^{mm})^2 \right. \\ & \quad + 2 \sum_{q \neq m} (\text{Re}\{N^{mm} N^{mq}\} (S \cos \omega(d_m - d_q) + \text{Re}\{N_{mq}^*\}) - \text{Im}\{N^{mm} N^{mq}\} (-S \sin \omega(d_m - d_q) + \text{Im}\{N_{mq}^*\})) \\ & \quad + 2 \sum_{p \neq m} \sum_{q > p, \neq m} (\text{Re}\{N^{pm} N^{mq}\} (S \cos \omega(d_p - d_q) + \text{Re}\{N_{pq}^*\}) - \text{Im}\{N^{pm} N^{mq}\} (-S \sin \omega(d_m - d_q) + \text{Im}\{N_{mq}^*\})) \\ & \quad + \sum_{p \neq m} |N^{pm}|^2 (S + N_1)] \\ & \quad + 2 \sum_m \sum_{n > m} [\text{Re}\{u_{mn}\} \langle \text{Re}\{G(m,n)\} \rangle - \text{Im}\{u_{mn}\} \langle \text{Im}\{G(m,n)\} \rangle] \end{aligned} \quad (55)$$

For the diagonals of the FIM, let

$k = i$, then

$$\left. \frac{\partial g_i}{\partial d_k} \right|_{k=i} = g_{ii} = -2\omega^2 \sum_{p \neq i} (\operatorname{Re}\{N^{ip}\} \cos\omega(d_i - d_p) - \operatorname{Im}\{N^{ip}\} \sin\omega(d_i - d_p)) \quad (56)$$

and

$$\frac{\partial^2 (V^* V^T)}{\partial d_i^2} = -\omega^2 \begin{pmatrix} & & & e^{j\omega(d_1 - d_i)} & & \\ & -0- & & e^{j\omega(d_2 - d_i)} & -0- & \\ & & & \vdots & & \\ e^{j\omega(d_i - d_1)} & e^{j\omega(d_i - d_2)} & \dots & 0 & \dots & e^{j\omega(d_i - d_M)} \\ & & & \vdots & & \\ & -0- & & e^{j\omega(d_M - d_i)} & -0- & \end{pmatrix} \quad (57)$$

These give

$$\frac{\partial^2 \Lambda'}{\partial d_i^2} = \sum_{B+} \frac{1}{g} \{ [g_{ii} - g_i^2/g] - X^T N^{-1} W N^{-1} X^* \} \quad (58)$$

where w_{mn} have the following values.

$m \neq i, n \neq i, m \neq n$

$$w_{mn} = (-g_{ii}/g) e^{j\omega(d_m - d_n)} + 2(g_i^2/g^2) e^{j\omega(d_m - d_n)} \quad (59a)$$

$m = i, n \neq i$ or $n = i, m \neq i$

$$w_{in} = (-\omega^2 - g_{ii}/g + 2 g_i^2/g^2 - 2(g_i/g)j\omega) e^{j\omega(d_i - d_n)} \quad (59b)$$

$$w_{mn} = w_{im}^*$$

$m = n$

$$w_{mm} = -g_{ii}/g + 2 g_i^2/g^2 \quad (59c)$$

Using the above results gives

$$(FIM)_{ii} = \sum_{B+} \frac{1}{g} (g_{ii} - g_i^2/g)$$

$$\begin{aligned}
& - \frac{1}{g} \sum_{B+} \left\{ \sum_m w_{mm} [(S+N_1)(N^{mm})^2 \right. \\
& + 2N^{mm} \sum_{q \neq m} ((S \cos \omega(d_m - d_q) + \operatorname{Re}\{N_{mq}^*\}) \operatorname{Re}\{N^{mq}\} \\
& \quad \left. - (-S \sin \omega(d_m - d_q) + \operatorname{Im}\{N_{mq}^*\}) \operatorname{Im}\{N^{mq}\}) \right. \\
& + 2 \sum_{p \neq m} \sum_{q > p, q \neq m} (S \cos \omega(d_p - d_q) + \operatorname{Re}\{N_{pq}^*\}) \operatorname{Re}\{N^{pm} N^{mq}\} \\
& \quad \left. - (-S \sin \omega(d_p - d_q) + \operatorname{Im}\{N_{pq}^*\}) \operatorname{Im}\{N^{pm} N^{mq}\}) \right. \\
& + \sum_{p \neq m} |N^{pm}|^2 (S + N_1) \left. \right\} \\
& + 2 \sum_m \sum_{n > m} [\operatorname{Re}\{w_{mn}\} < \operatorname{Re}\{G(m, n)\} > \\
& \quad - \operatorname{Im}\{w_{mn}\} < \operatorname{Im}\{G(m, n)\} >] \quad (60)
\end{aligned}$$

To compare with previous results observe in (55) and (60) that if noise is spatially uncorrelated, $g_i = 0$, and only $u_{ik} = u_{ki}^* = \omega^2 e^{\pm j\omega(d_i - d_k)}$ and $w_{in} = w_{ni}^* = -\omega^2 e^{\pm j\omega(d_i - d_n)}$ are non-zero. Further, in (52) and (53) $p = m$ and $q = n$ are the only non-zero terms. Utilizing the above,

$$\begin{aligned}
(\text{FIM})_{ik} &= -2 \sum_{B+} \frac{\omega^2}{g} (\cos \omega(d_i - d_k) N^{ii} N^{kk} S \cos \omega(d_i - d_k) \\
& \quad + \sin \omega(d_i - d_k) N^{ii} N^{kk} S \sin \omega(d_i - d_k)) \\
&= -2 \sum_{B+} \frac{\omega^2}{g} N^{ii} N^{kk} S,
\end{aligned}$$

$$\text{where } g = \frac{1}{S} + \sum_i N^{ii},$$

and similarly

$$(\text{FIM})_{ii} = 2 \sum_{B+} \sum_{n \neq i} \frac{\omega^2}{g} N^{ii} N^{nn} S.$$

It is readily seen that this FIM is identical to Eq. 7 (the same as Eq. 12 in [3]).

Unfortunately the FIM defined by (55) and (60) has elements which are in general functions of the delays themselves, making analysis difficult. However, in the next section we will assume a signal source at infinity, allowing some simplification.

VII. Evaluation of the CRMB

It is unreasonable to evaluate and invert the FIM in sections IV or VI in general because it is a function of all d_i . However, if wavefront curvature is ignored, each delay may be written $d_i = i\Delta$ where Δ is the delay between adjacent sensors. We may also let Δ vary between zero and $\omega\Delta = \pi$ for a single frequency. Then $d_p - d_q = (p-q)\Delta$ for example. This is the beam former case.

Because of the generality of the formulas we may also vary the elements of N , using the symmetric matrix (as in [11])

$$N = N_1 \begin{pmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{m-1} \\ & 1 & \rho_1 & \cdots & \rho_{m-2} \\ & & 1 & \cdots & . \\ & & & \ddots & . \\ & & & & 1 \end{pmatrix}$$

wherein $\rho_r = \rho_0 e^{-r\sigma} e^{j\omega\beta}$. Such a correlation is appropriate for turbulent boundary layer noises and its magnitude with respect to the unity diagonal accounts for isotropic noise. In the following simulations we choose $|\rho_1| = 0, 0.2, 0.4$ and $\omega\beta$ having values 0 through $\pi/2$.

Figures 1-24 show the CRMB (1,1) element, center element, or last element as a function of the various parameters. Table I is presented as a guide to comparisons.

The formulas for the FIM may be applied to arrays with clustered elements as well, if the spacing between clusters is considered. We have done this in producing the data in Figures 25 through 30. Zero correlation between clusters is assumed.

The clustered (or grouped) arrays studied are as shown here.

.	.	.	← 3 sensors
...	← 9 sensors
.....	← 15 sensors

The spacing between array ends and ends-to-center remains fixed. The effect of adding sensors to the cluster when spatially correlated noise is present can then be observed.

Comments derived from the Figures are as follows

Table I. Guide to Figure Variables

Fig #	Sensors M	Measurements	SNR	ℓ	$\omega\Delta$	Ordinate CRMB element	Abscissa	Notes
1	3	2		6 values	0	1,1	SNR	
2	3	2		"	0	2,2	SNR	
3	3	2		"	0.7854	1,1	SNR	
4	3	2		"	0.7854	2,2	SNR	
5	3	2		"	1.571	1,1	SNR	
6	3	2		"	1.571	2,2	SNR	
7	15	14		"	0	1,1	SNR	
8	15	14		"	0	7,7	SNR	
9	15	14		"	0	14,14	SNR	
10	3	2	0.1	0.0,0.2,0.4		1,1	$\omega\Delta$	
11	3	2	0.1	"		2,2	$\omega\Delta$	
12	3	2	1.0	"		1,1	$\omega\Delta$	
13	3	2	1.0	"		2,2	$\omega\Delta$	
14	3	2	10.0	"		1,1	$\omega\Delta$	
15	3	2	10.0	"		2,2	$\omega\Delta$	
16	15	14	0.1	"		1,1	$\omega\Delta$	
17	15	14	0.1	"		7,7	$\omega\Delta$	
18	15	14	0.1	"		14,14	$\omega\Delta$	
19	15	14	1.0	"		1,1	$\omega\Delta$	
20	15	14	1.0	"		7,7	$\omega\Delta$	
21	15	14	1.0	"		14,14	$\omega\Delta$	
22	15	14	10.0	"		1,1	$\omega\Delta$	
23	15	14	10.0	"		7,7	$\omega\Delta$	
24	15	14	10.0	"		14,14	$\omega\Delta$	
25	9	8		0.2	0	8,8	SNR	} equi-spaced vs grouped array
26	9	8		0.2	0	4,4	SNR	
27			1.0	0.0,0.2,0.4	0	1,1	sensors,M	
28			0.1,1.0,10	0.2	0	1,1	sensors,M	"
29			0.1,1.0,10	0.2	0	M-1,M-1	sensors,M	grouped array
30			1.0	0.0,0.2,0.4	0	M-1,M-1	sensors,M	"

1. Figures 1 through 9 show that variance decreases monotonically with SNR and that variations in ρ from 0 through 0.4 45° have unpredictable, but not large effects.
2. Figures 1 and 7 for example show that more sensors (from 3 to 15) will reduce the variance of a delay.
3. Figures 1,2 or 3,4 or 5,6 or 7,8,9 show that variance bounds for delays end-to-center will vary with ρ differently than those for end-to-end, but not a lot. Also the end-to-end delays vary least.
4. Comparing Figs. 1 and 3 for example shows that the effect of ρ on a delay estimator will vary with $\omega\Delta$ (look angle.) This variance is easier to see in Figures 10-24.
5. Figures 10-24 demonstrate that the bounds are effected by look angle to a much larger extent when ρ is increased to 0.4. As much as 5dB (Figs. 11,17) is observed at SNR = 0.1.
6. Comparing Figures in 10-24 with like SNR and M shows that different delays are effected quite differently as $\omega\Delta$ varies; i.e. CRMB(1,1), (2,2), (7,7) or (14,14) all vary differently with ρ and $\omega\Delta$.
7. Grouping sensors when spatial noise correlation is present has a detrimental effect at low SNR. This may be seen in Figures 25, 26, which also show that the midpoint and end delay variances are equal at $\omega\Delta = 0$. (They are not equal at other $\omega\Delta$ per comment 4 above). The difference between curves A and B is that the 9 x 9 noise covariance matrix for curve B has 3 x 3 blocks on the diagonal while curve A's noise matrix is full. Thus curve A represents a cluster of nine sensors while curve B assumes each cluster of 3 has noise independent from the other clusters. For $\rho = 0.2$ and $\omega\Delta = 0$ the effect is 0.3 dB at SNR = 0.1.
8. Pursuing the question of how much clustering is effective when spatial noise is present, Figures 27 through 30 plot the variance bounds vs sensor number M while holding array length constant. We conclude that delay variances are reduced much less for M changing from 9 to 15 than for M changing from 3 to 9.

The last comment is meant to be one of the basic conclusions of this study: that for significant spatially correlated noise, there is a point beyond which it does not pay to increase sensor number in a cluster when the purpose is to reduce delay variance between clusters.

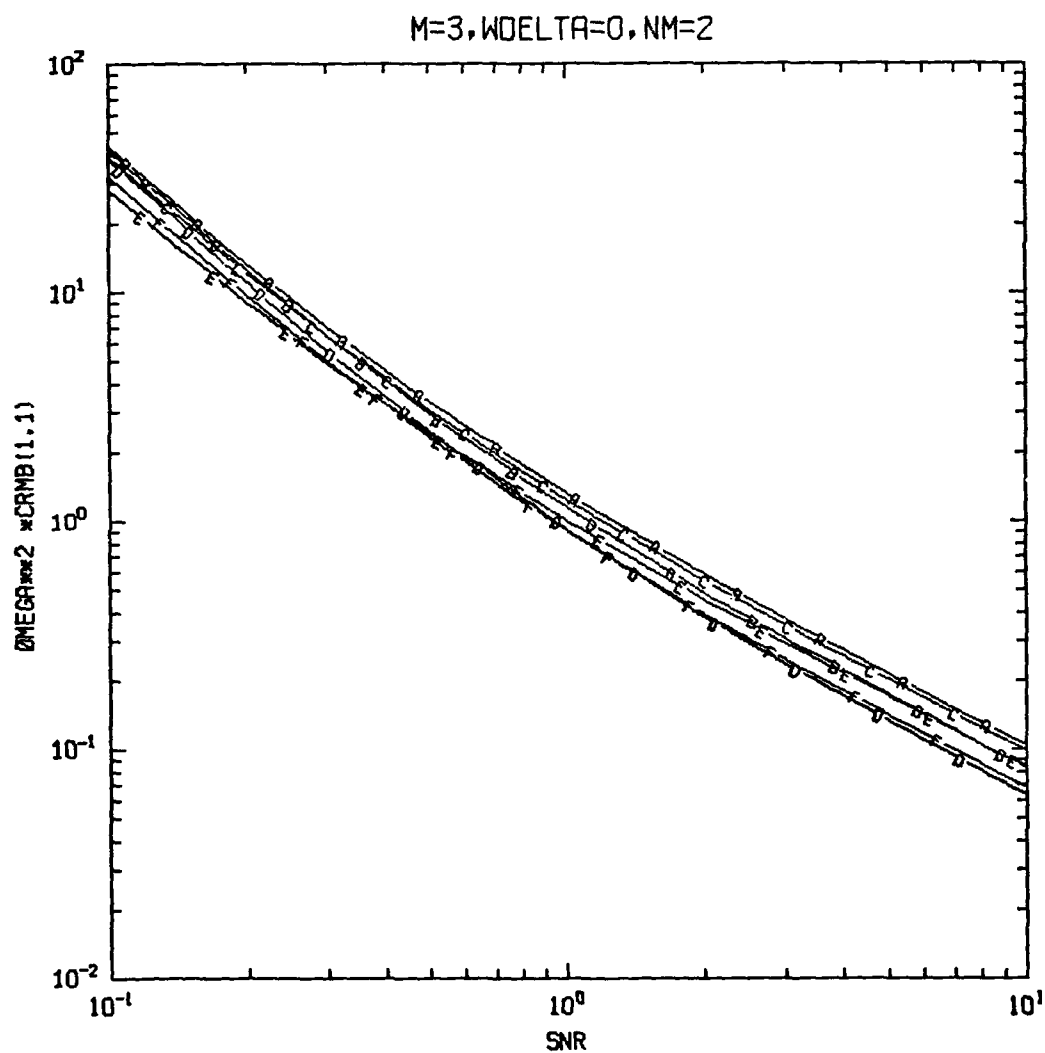


Figure 1. CRMB(1,1) vs SNR. $M=3$. A-- $\rho=0$, B-- $\rho=0.2$, C-- $\rho=j0.2$;
D-- $\rho=0.4$, E-- $\rho=j0.4$, F-- $\rho=0.4(1+j)$. $\omega\Delta=0$

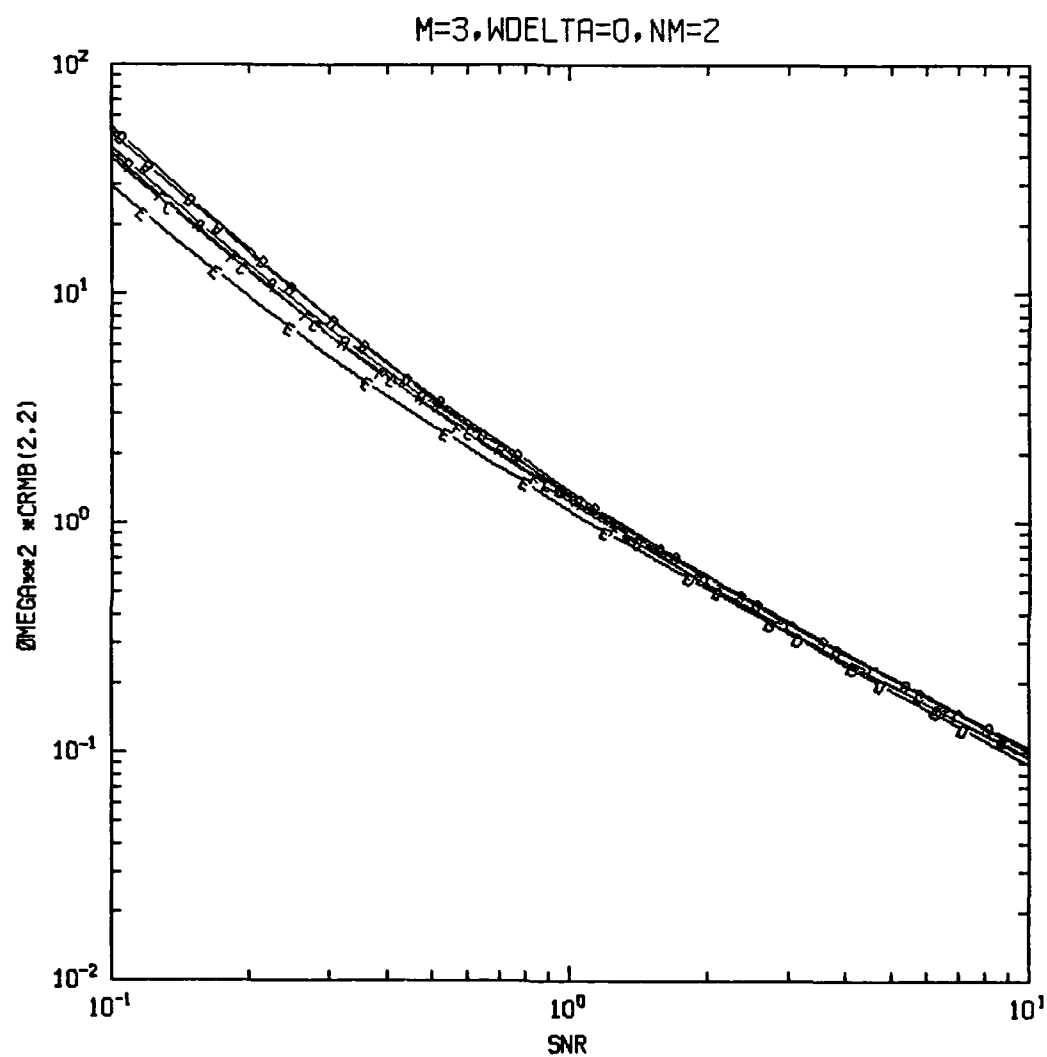


Figure 2. CRMB (2,2) vs SNR, $M=3$. A-- $\rho=0$, B-- $\rho=0.2$, C-- $\rho=j0.2$;
D-- $\rho=0.4$, E-- $\rho=j0.4$, F-- $\rho=0.4(1+j)$. $\omega\Delta=0$

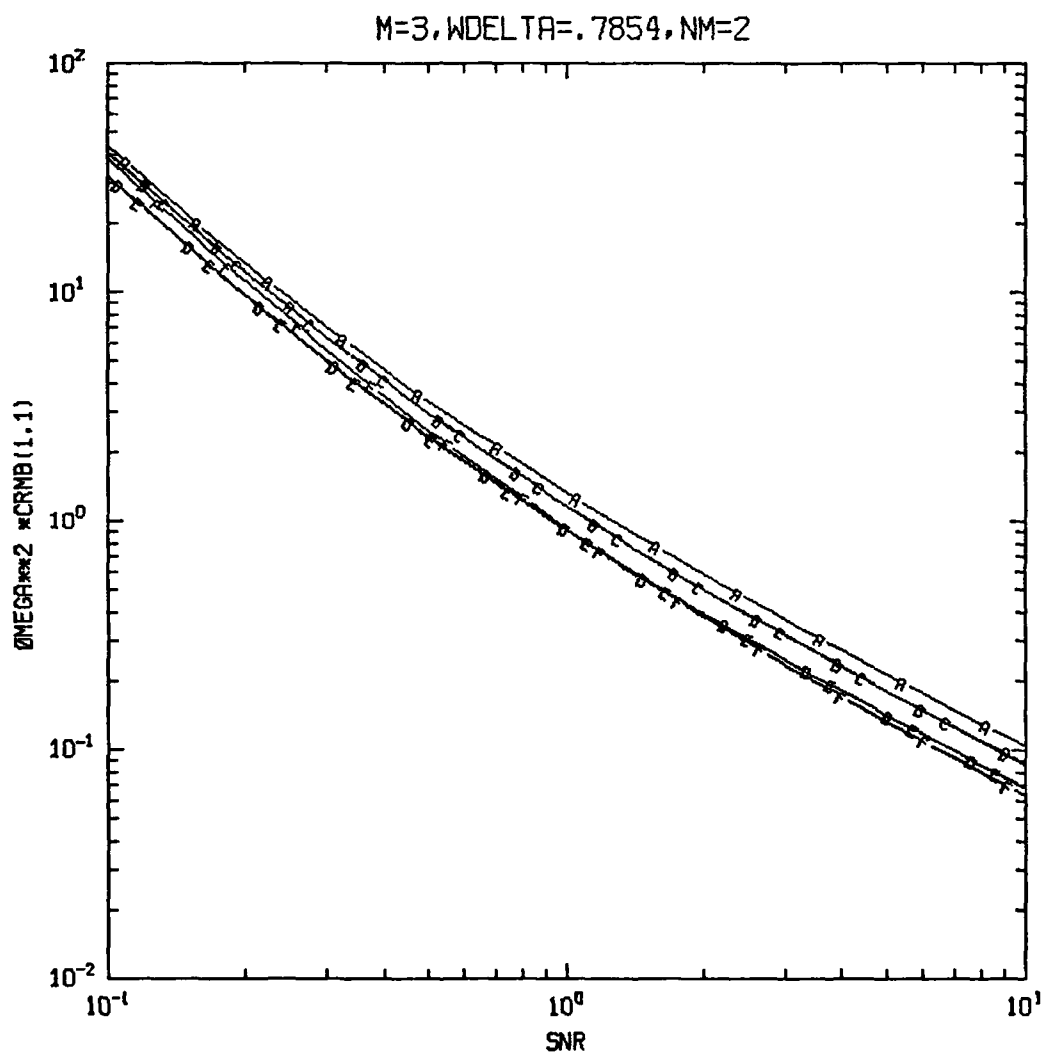


Figure 3. $\text{CRMB}(1,1)$ vs SNR. $M=3$. A-- $\rho=0$, B-- $\rho=0.2$, C-- $\rho=j0.2$;
D-- $\rho=0.4$, E-- $\rho=j0.4$, F-- $\rho=0.4(1+j)$. $\omega\Delta=\pi/4$

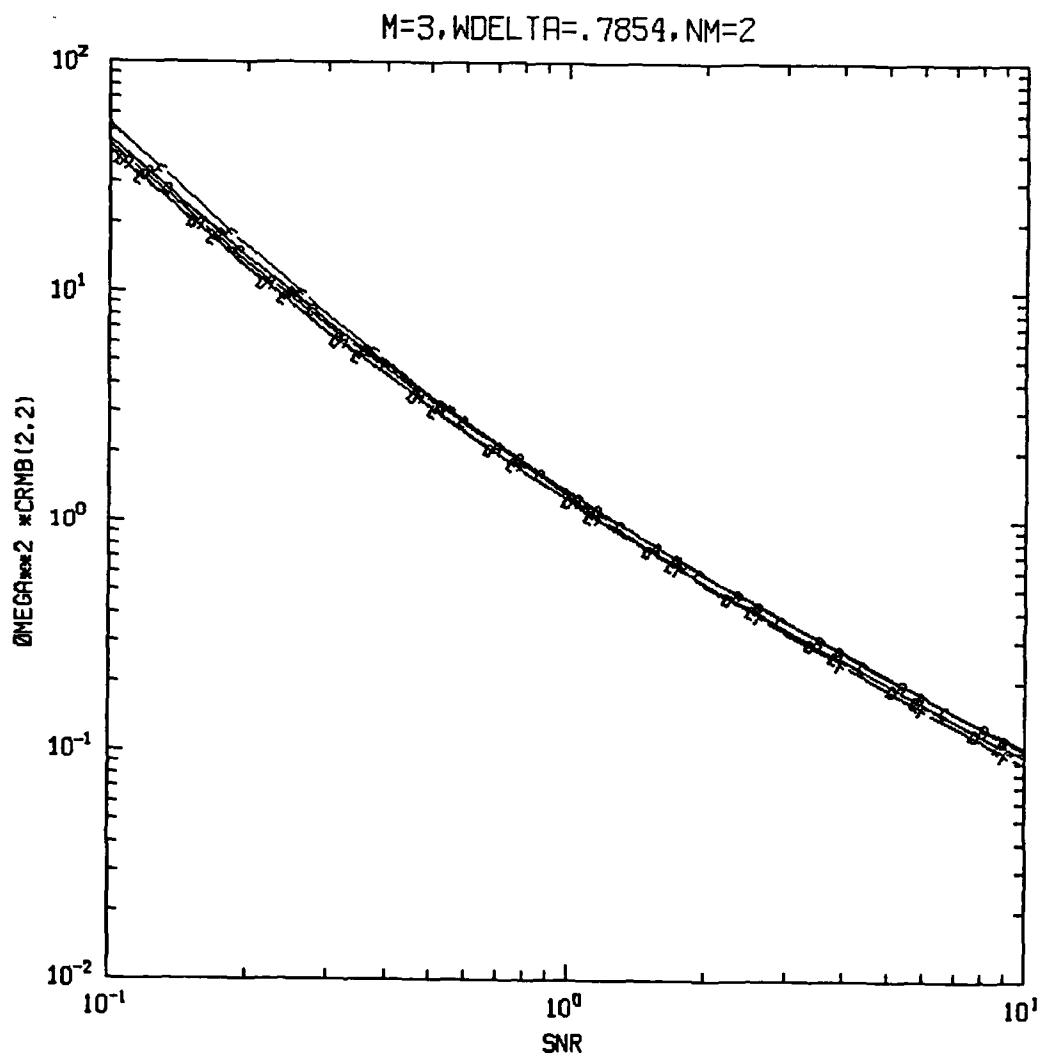


Figure 4. CRMB(2,2) vs SNR. $M=3$. A-- $\rho=0$, B-- $\rho=0.2$, C-- $\rho=j0.2$;
D-- $\rho=0.4$, E-- $\rho=j0.4$, F-- $\rho=0.4(1+j)$. $\omega\Delta=\pi/4$

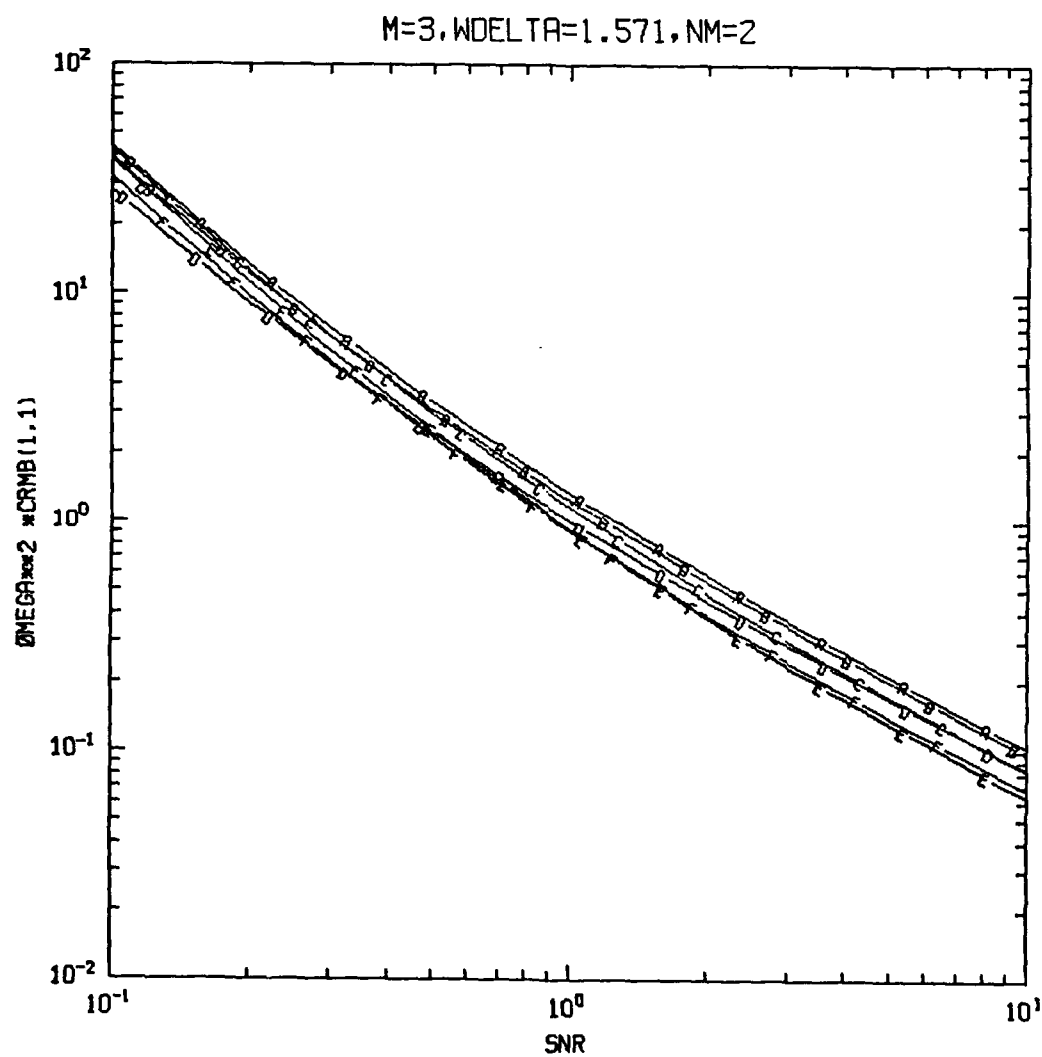


Figure 5. $CRMB(1,1)$ vs SNR. $M=3$. A-- $\rho=0$, B-- $\rho=0.2$, C-- $\rho=j0.2$;
D-- $\rho=0.4$, E-- $\rho=j0.4$, F-- $\rho=0.4(1+j)$. $\omega\Delta T = \pi/2$

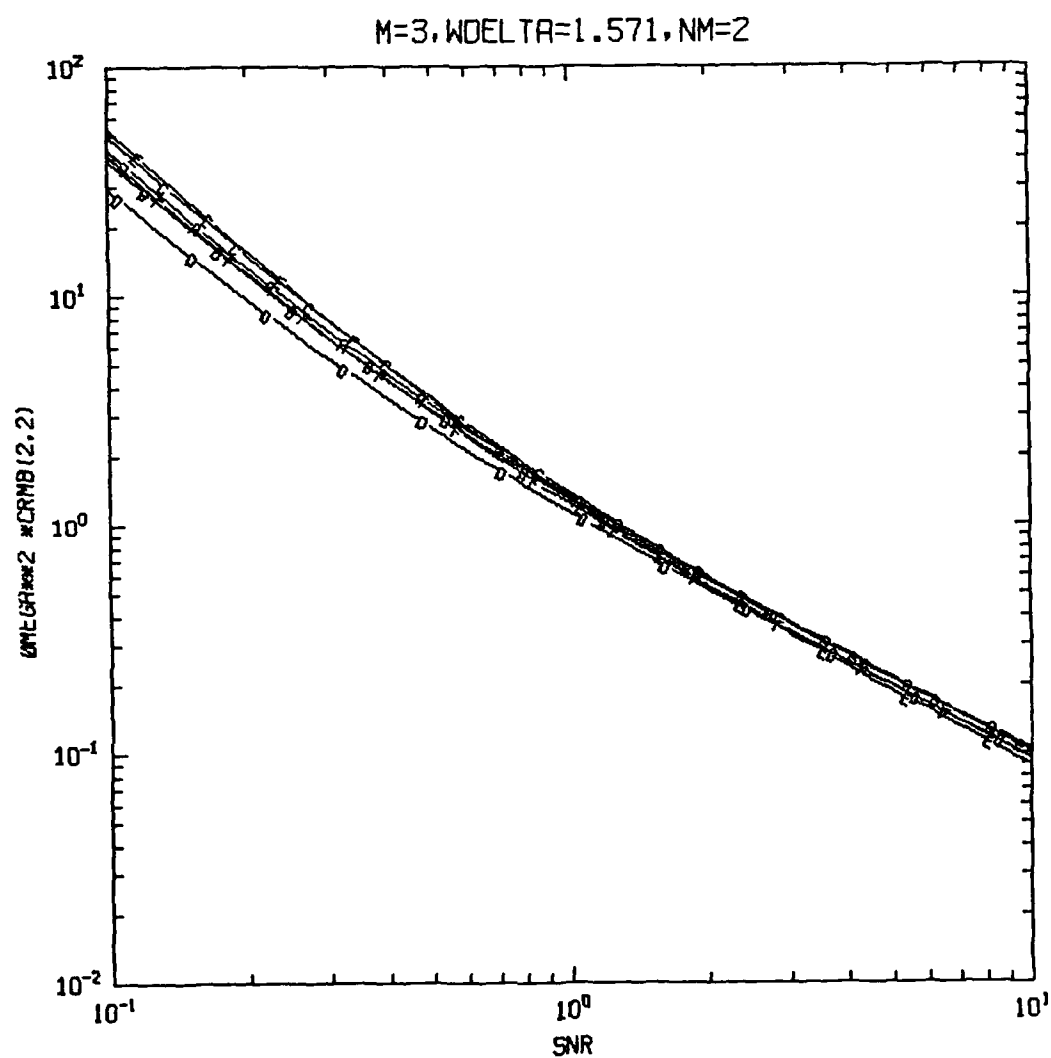


Figure 6. CRMB(2,2) vs SNR. $M=3$. A-- $\rho=0$, B-- $\rho=0.2$, C-- $\rho=j0.2$;
D-- $\rho=0.4$, E-- $\rho=j0.4$, F-- $\rho=0.4(1+j)$. $\omega\Delta t=\pi/2$

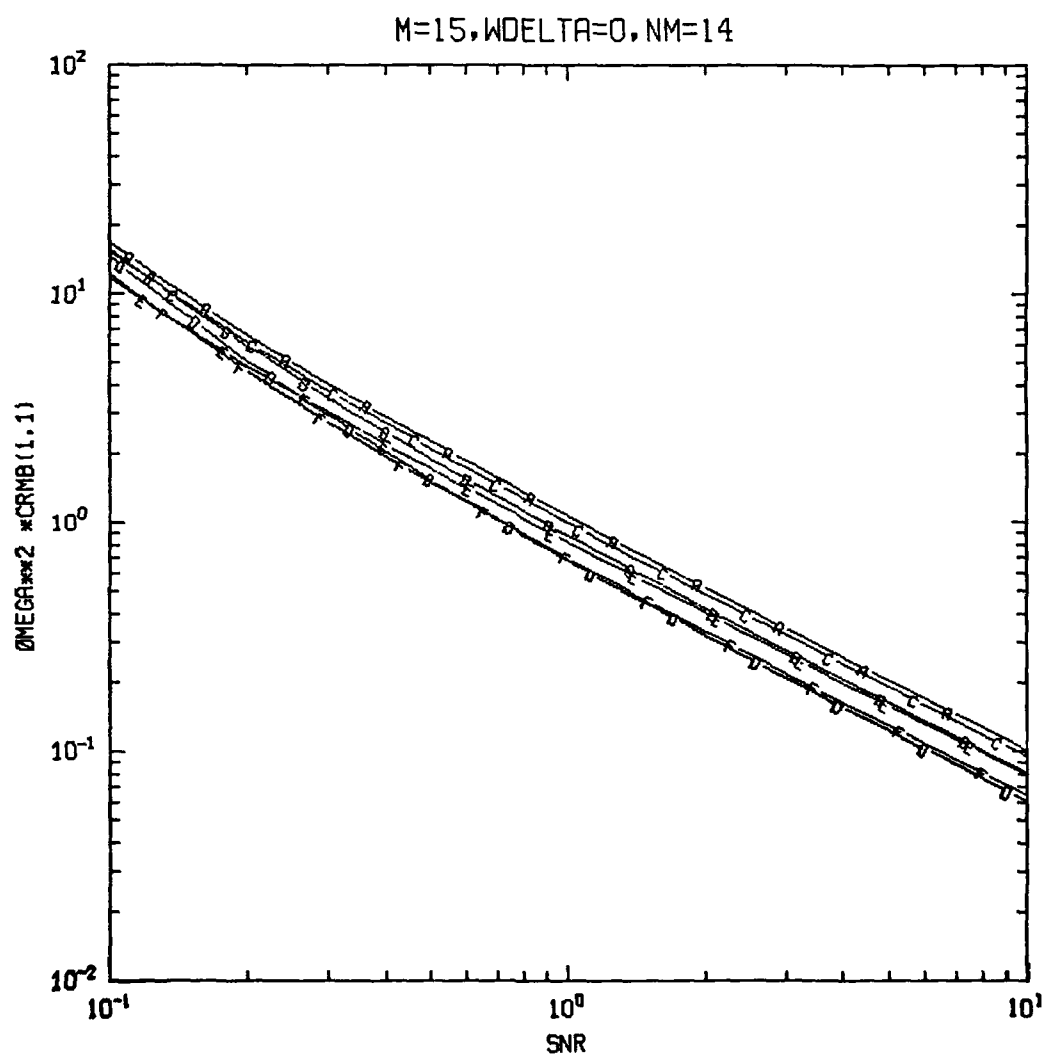


Figure 7. CRMB(1,1) vs SNR. $M=15$. A-- $\rho=0$, B-- $\rho=0.2$, C-- $\rho=j0.2$;
D-- $\rho=0.4$, E-- $\rho=j0.4$, F-- $\rho=0.4(1+j)$. $\omega\Delta=0$

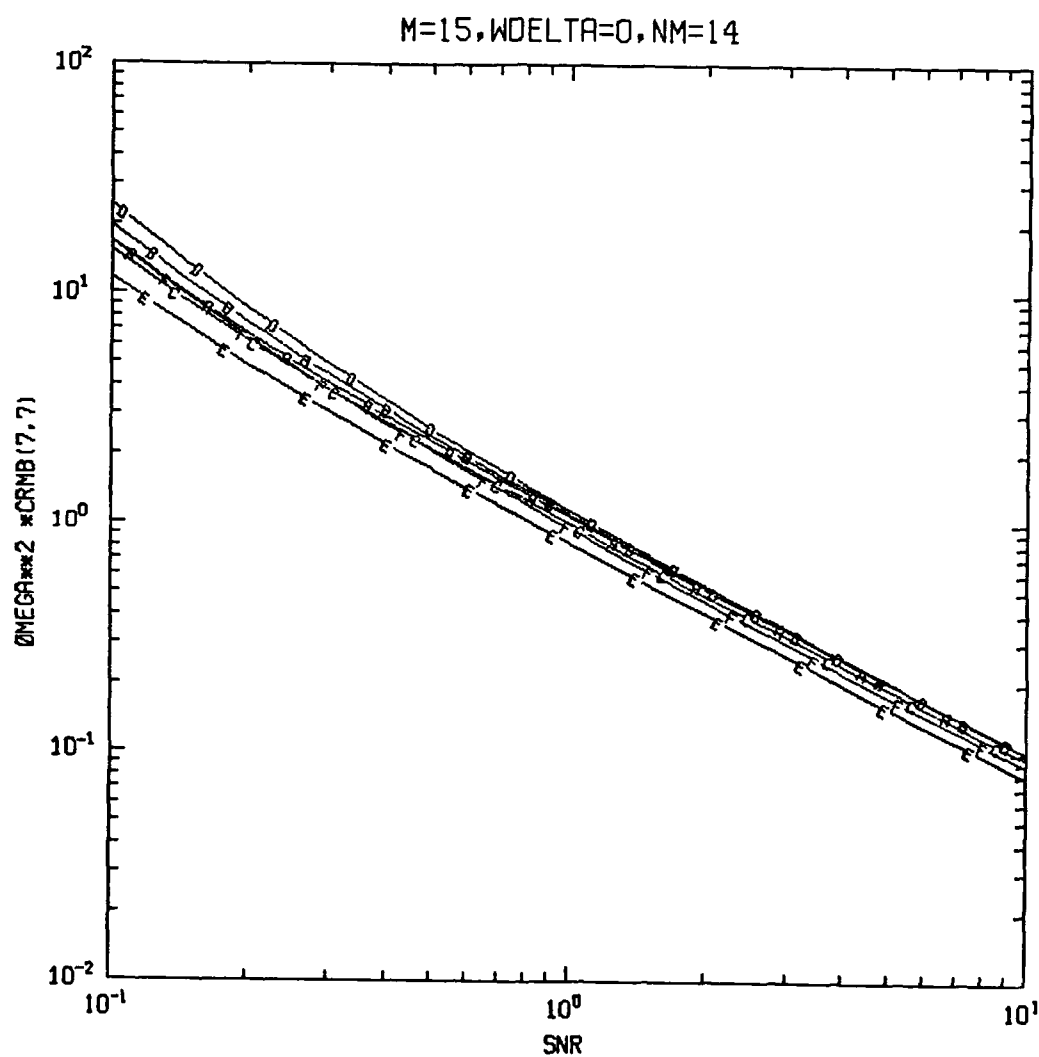


Figure 8. $CRMB(7,7)$ vs SNR . $M=15$. A-- $\rho=0$, B-- $\rho=0.2$, C-- $\rho=j0.2$;
D-- $\rho=0.4$, E-- $\rho=j0.4$, F-- $\rho=0.4(1+j)$. $\omega\Delta=0$

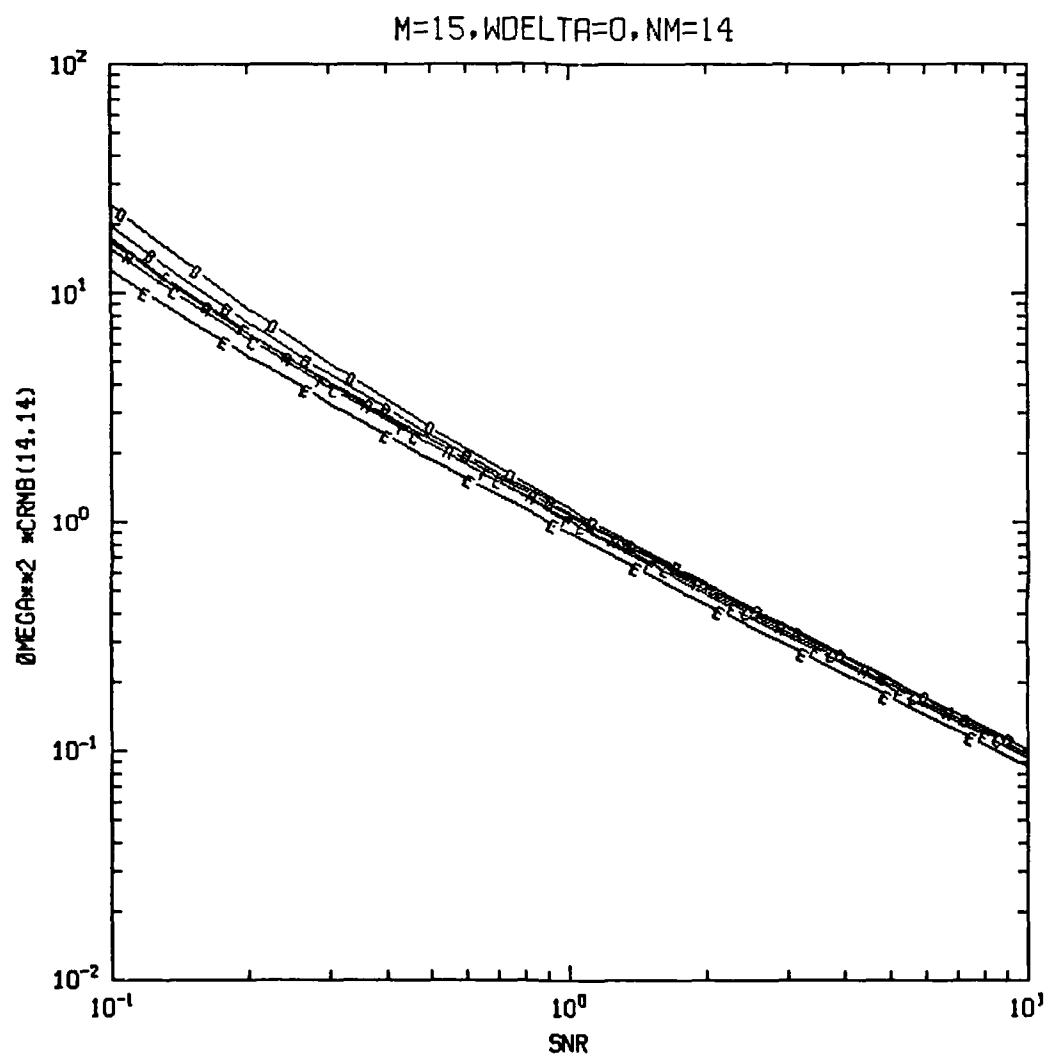


Figure 9. CRMB(14,14) vs SNR. $M=15$. A-- $\rho=0$, B-- $\rho=0.2$, C-- $\rho=j0.2$;
D-- $\rho=0.4$, E-- $\rho=j0.4$, F-- $\rho=0.4(1+j)$. $\omega\Delta=0$

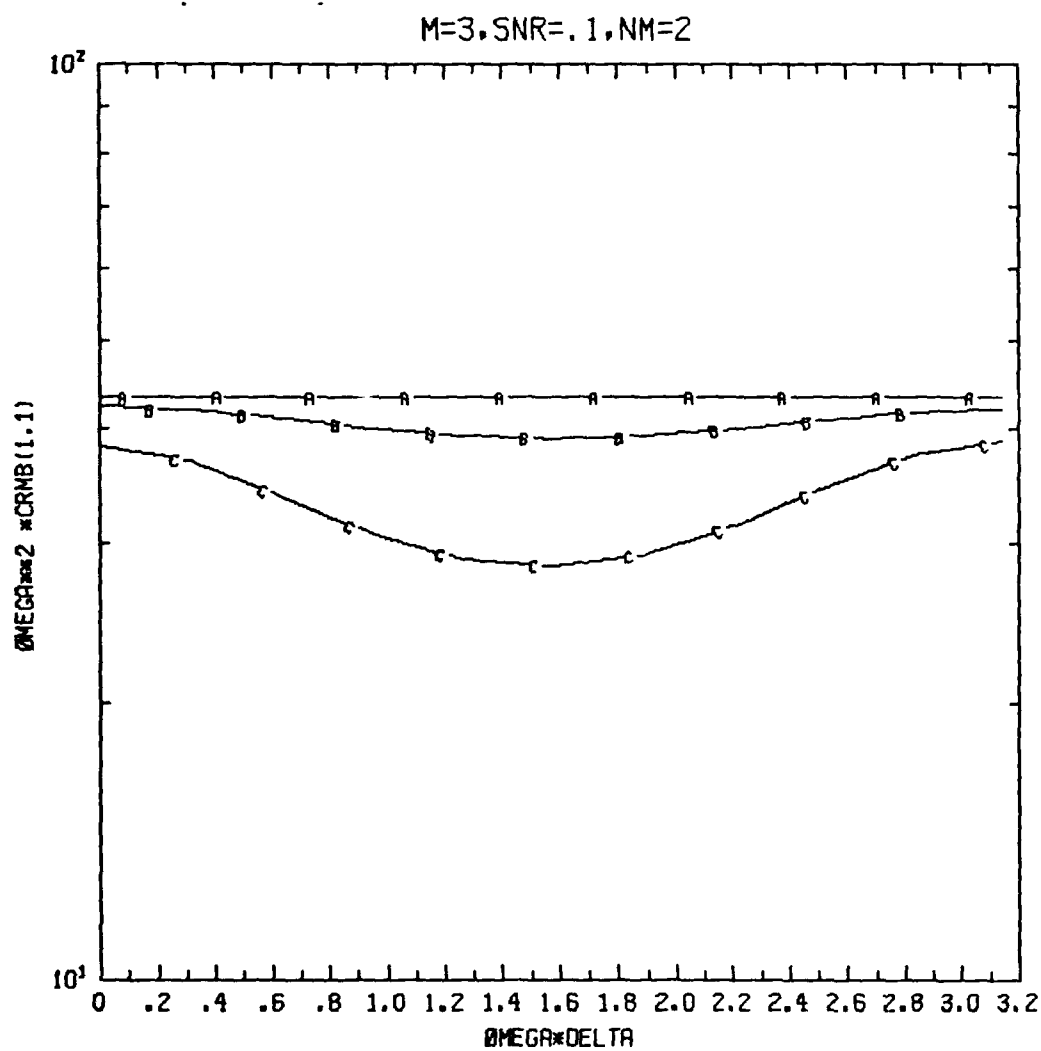


Figure 10. CRMB(1,1) vs $\omega\Delta$. $M=3$. A-- $\rho=0.0$, B-- $\rho=0.2$;
C-- $\rho=0.4$. SNR=0.1

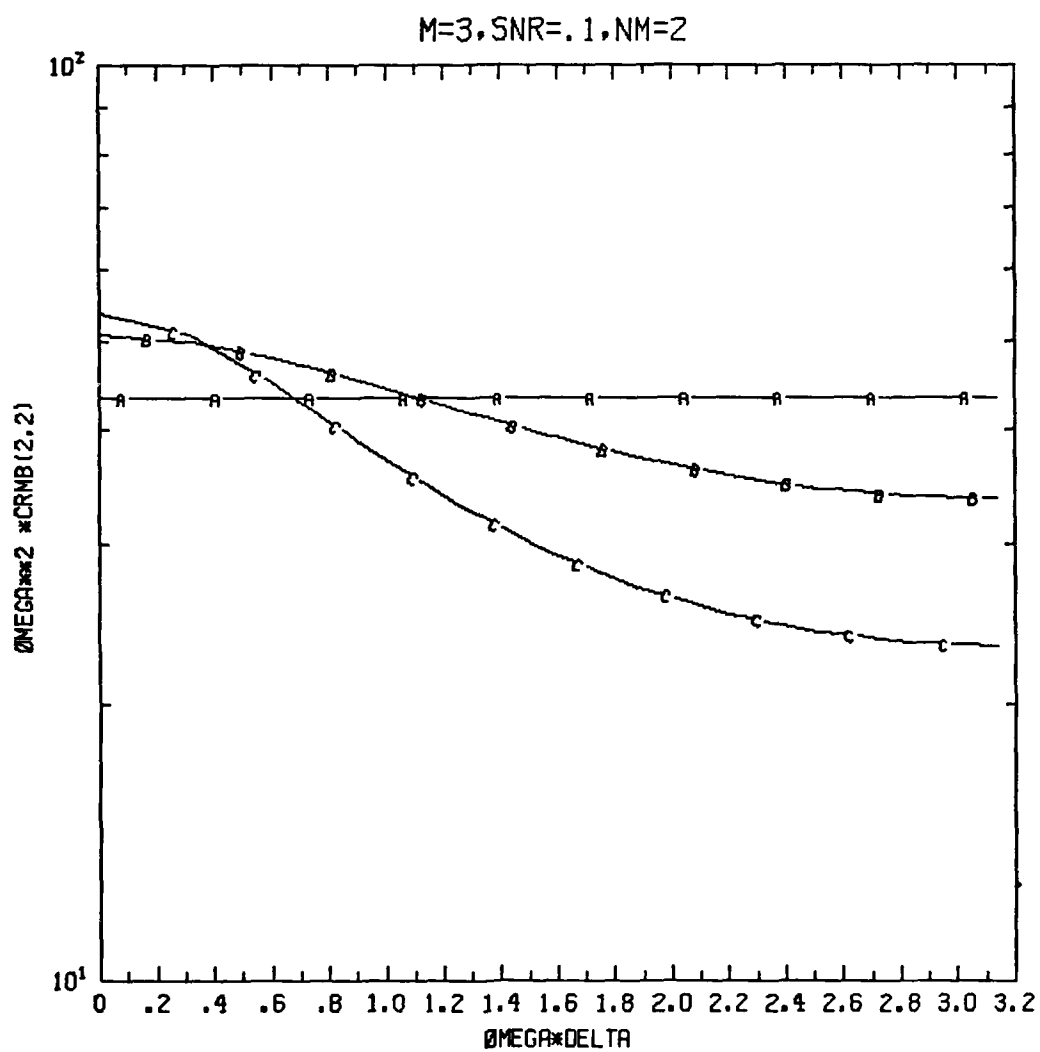


Figure 11. $CRMB(2,2)$ vs $\omega\Delta$. $M=3$. A-- $\rho=0.0$, B-- $\rho=0.2$;
C-- $\rho=0.4$. $SNR=0.1$

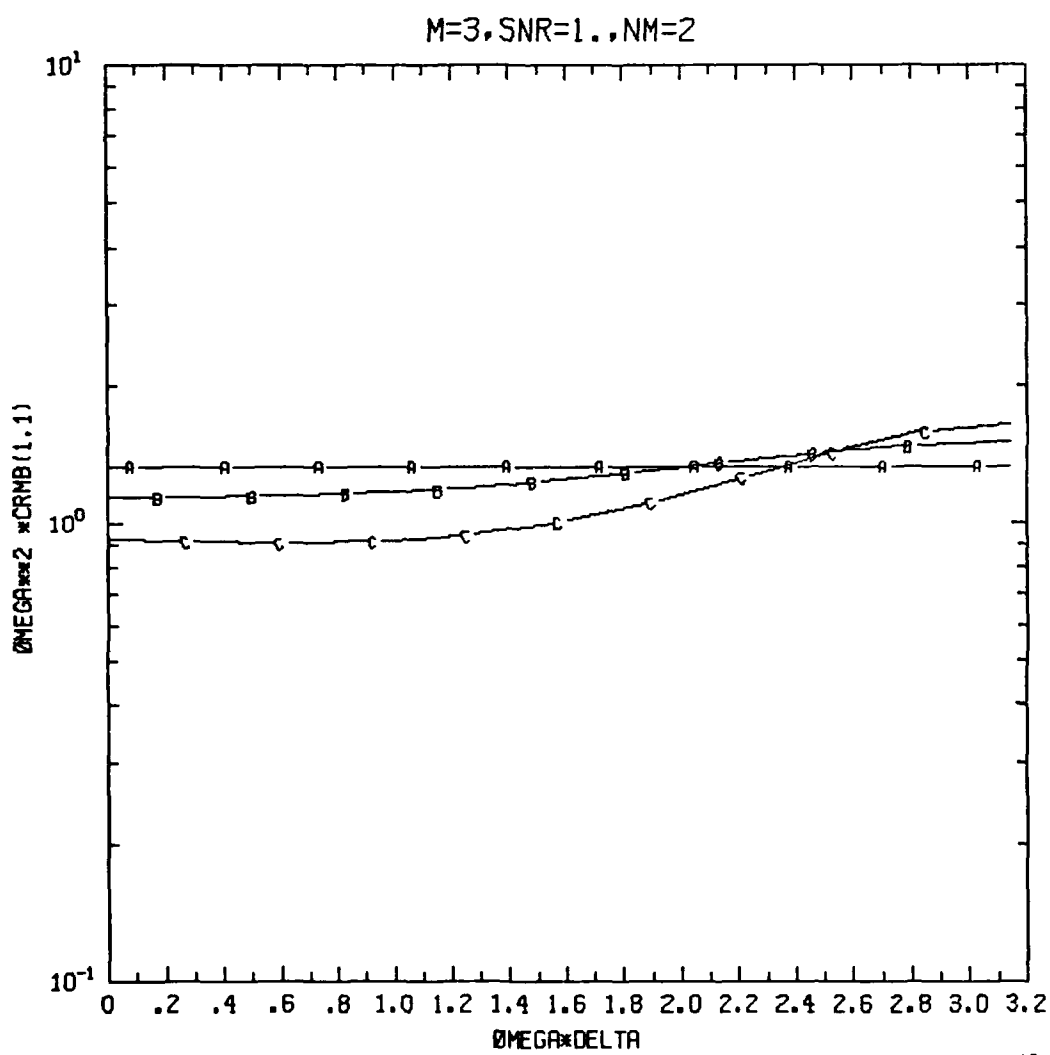


Figure 12. CRMB(1,1) vs $\omega\Delta$. $M=3$. A-- $\rho=0.0$, B-- $\rho=0.2$;
C-- $\rho=0.4$. SNR=1.0

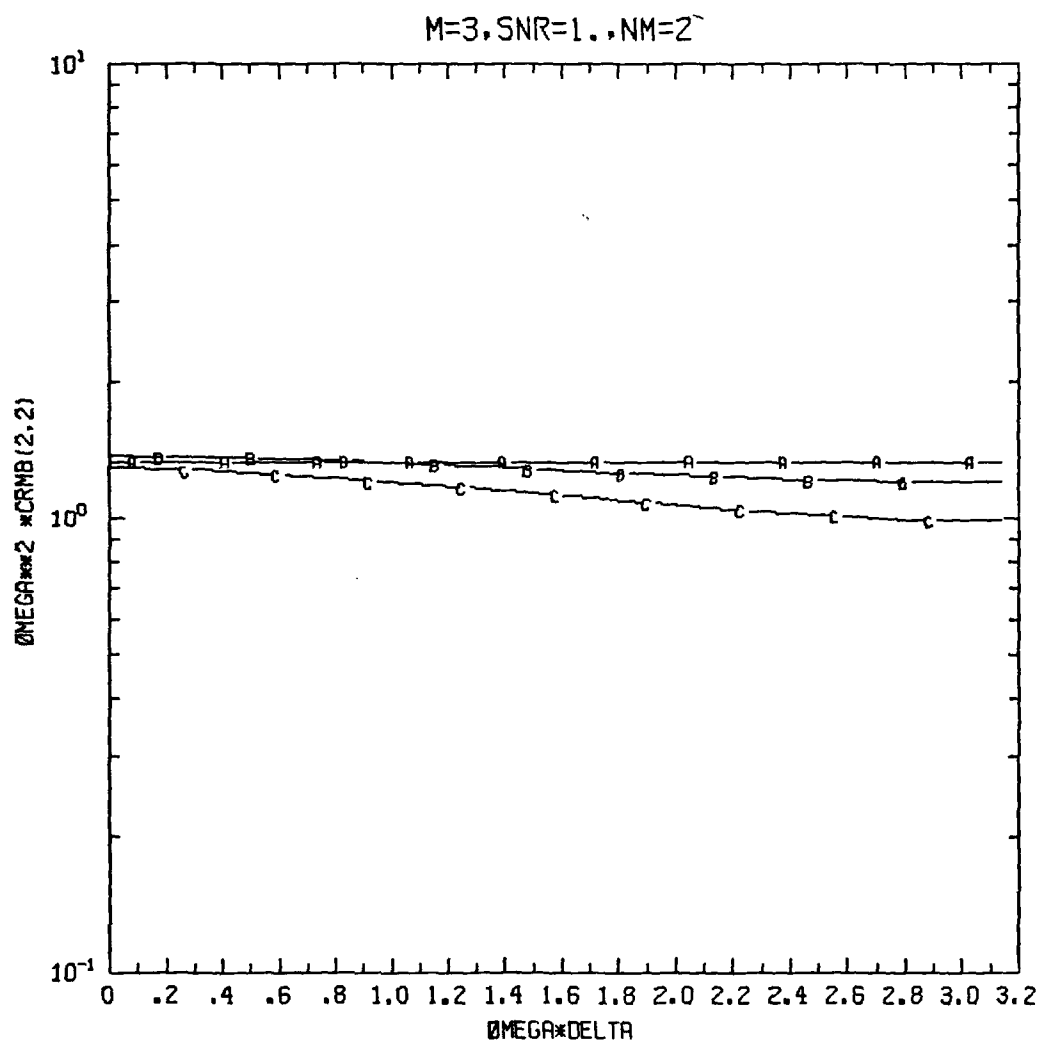


Figure 13. CRMB(2,2) vs $\omega\Delta$. $M=3$. A-- $\rho=0.0$, B-- $\rho=0.2$;
C-- $\rho=0.4$. SNR=1.0

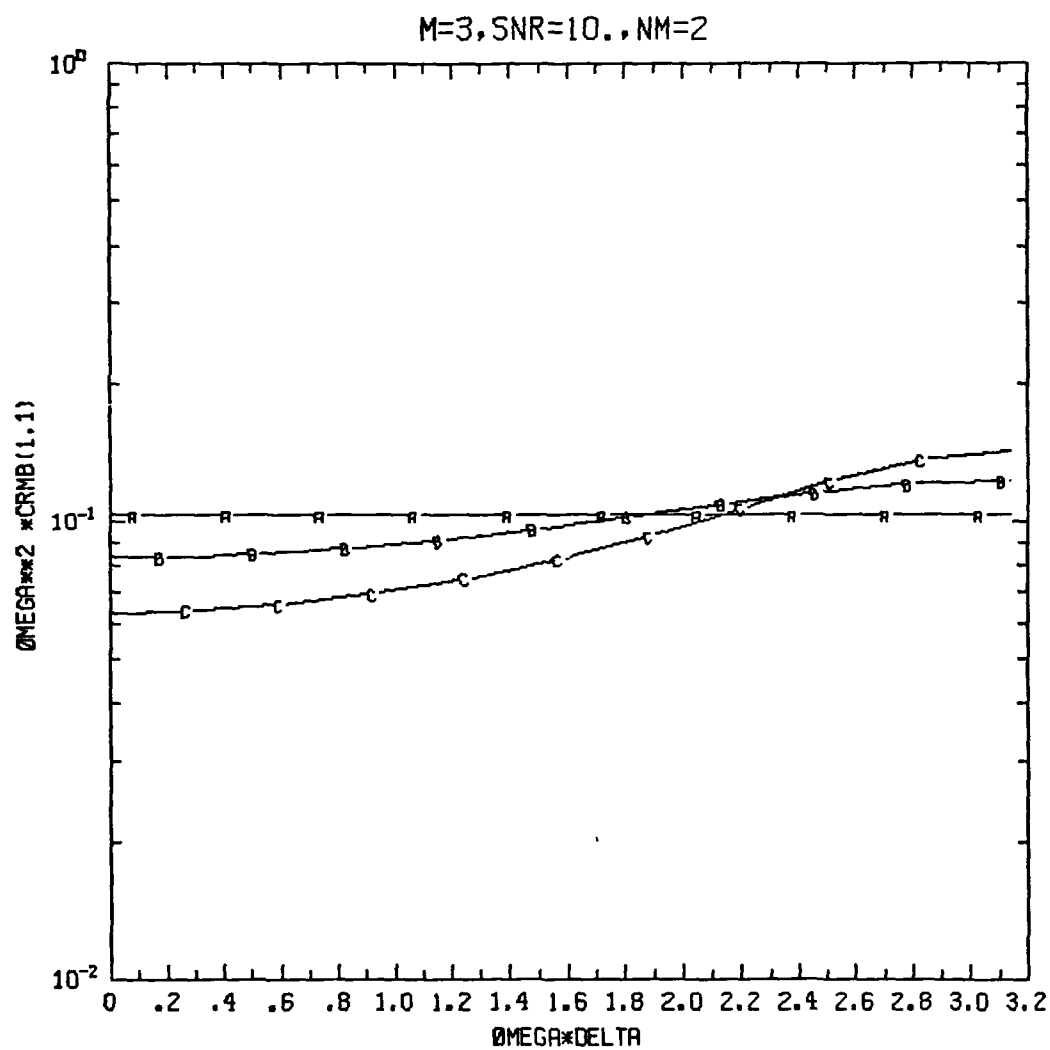


Figure 14. CRMB(1,1) vs $\omega \Delta$. $M=3$. A-- $\rho=0.0$, B-- $\rho=0.2$;
C-- $\rho=0.4$. SNR=10.0

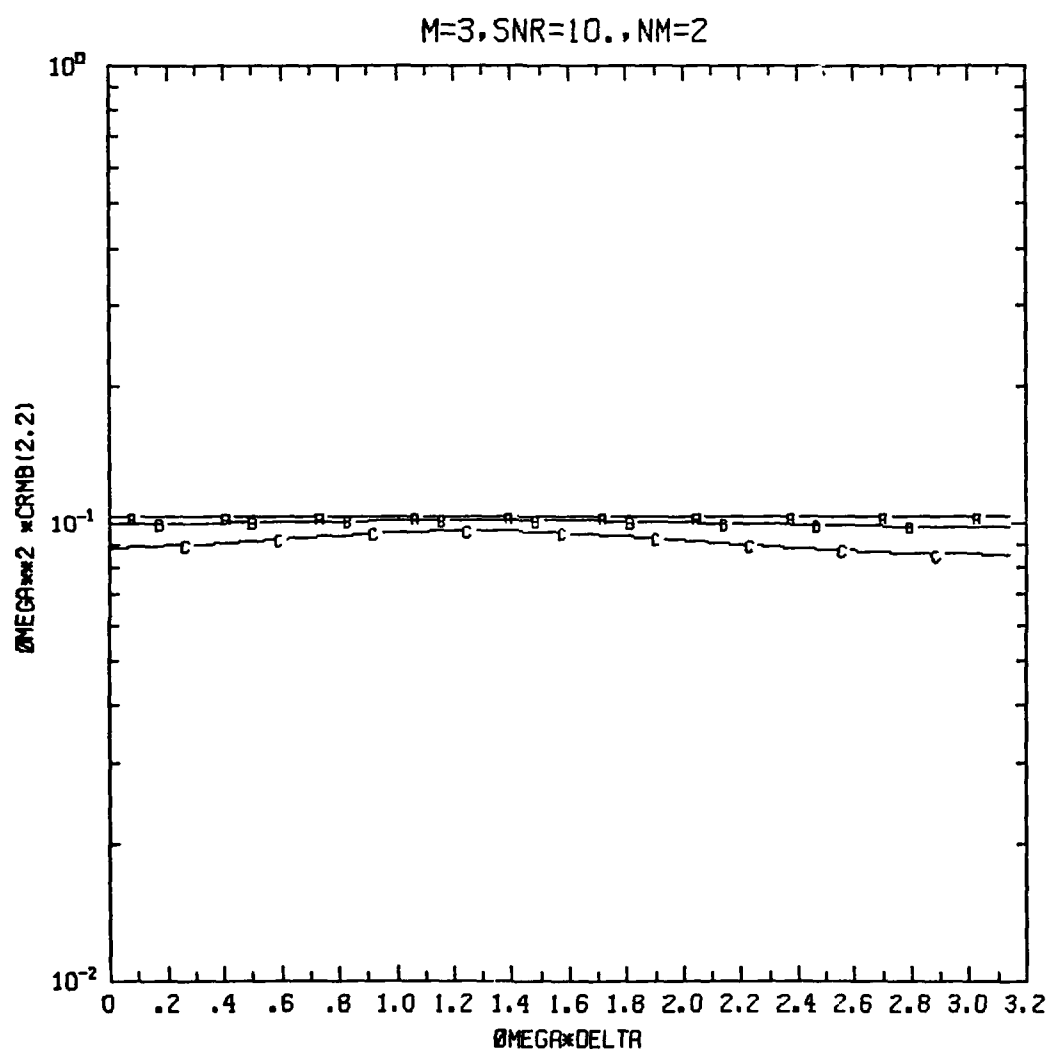


Figure 15. CRMB(2,2) vs $\omega\Delta$. $M=3$. A-- $\rho=0.0$, B-- $\rho=0.2$;
C-- $\rho=0.4$. SNR=10.0

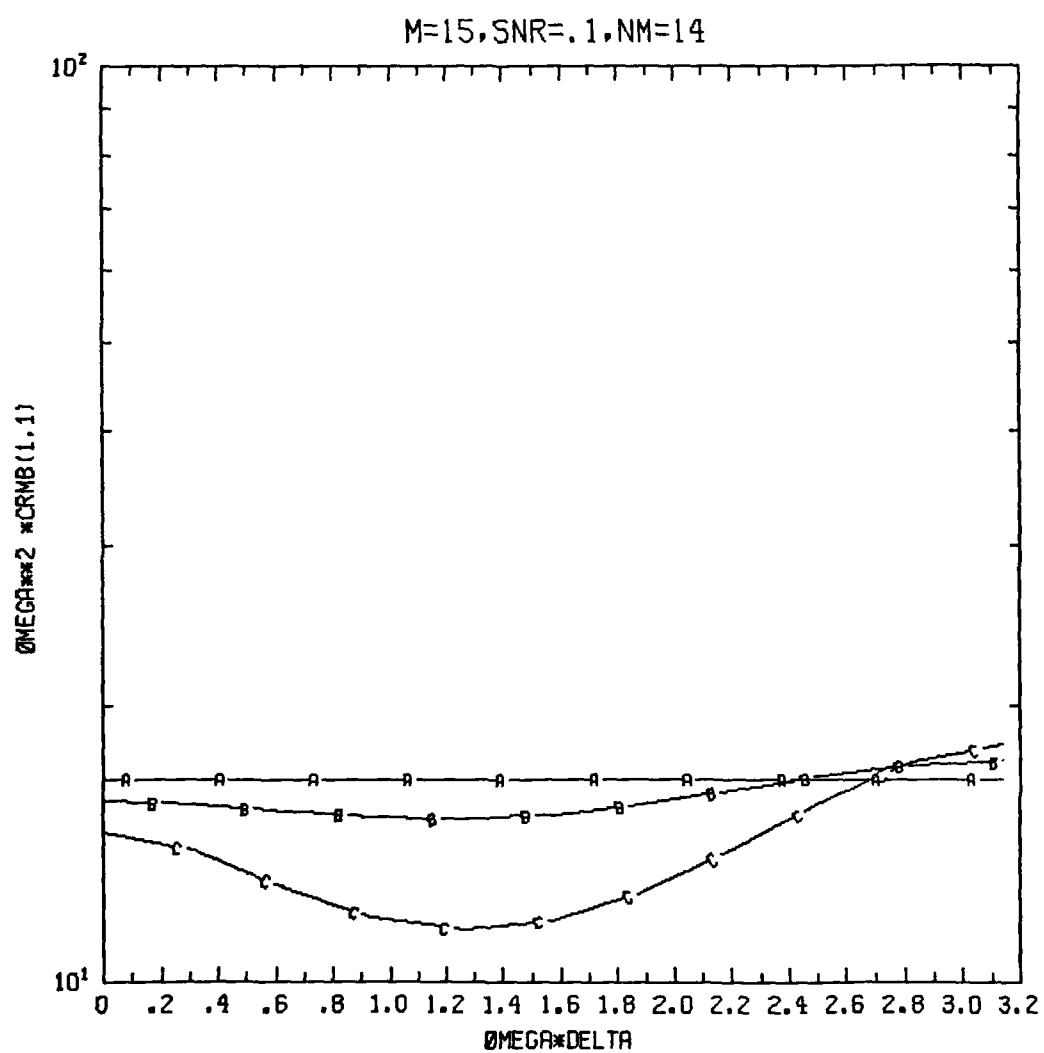


Figure 16. $\text{CRMB}(1,1)$ vs $\omega \Delta$. $M=15$. A-- $\rho=0.0$, B-- $\rho=0.2$;
C-- $\rho=0.4$. $\text{SNR}=0.1$

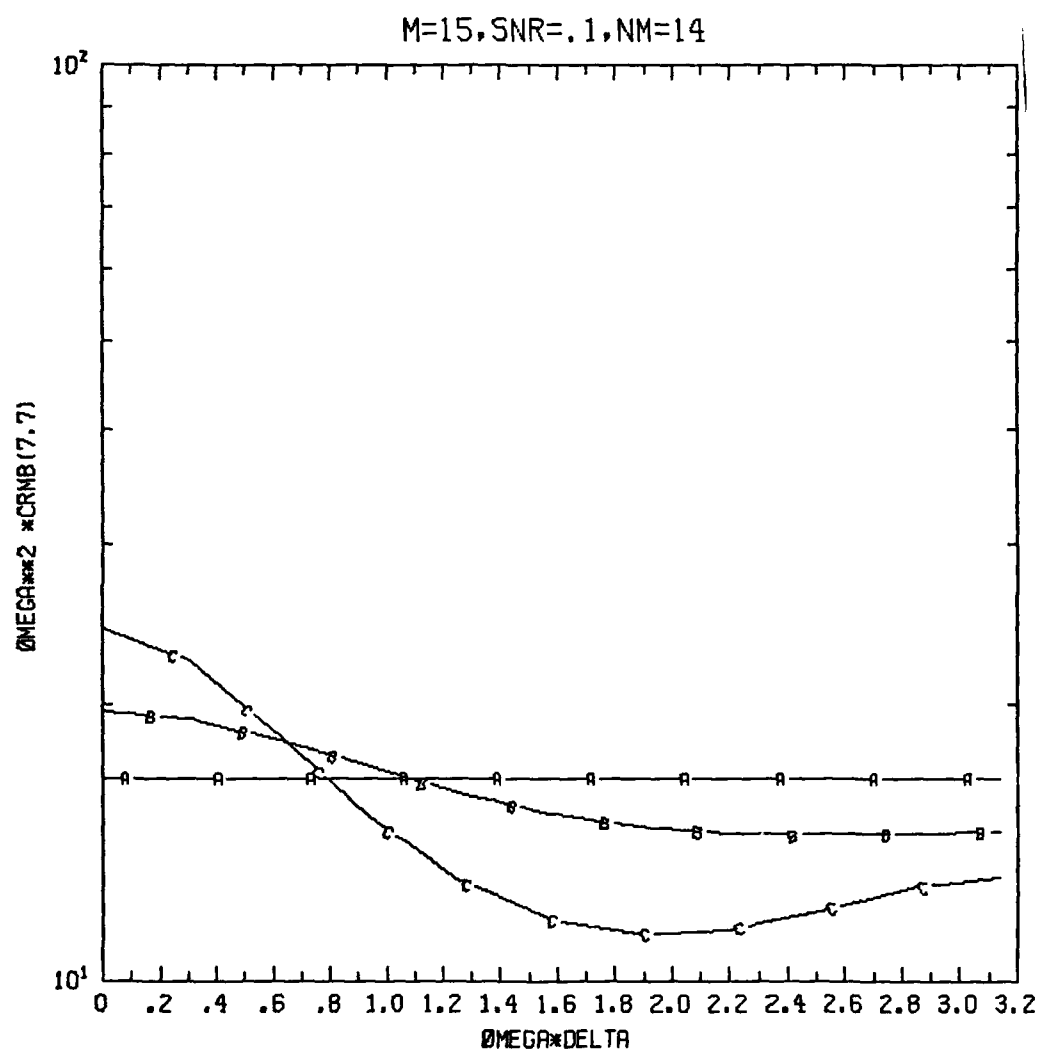


Figure 17. CRMB(7,7) vs $\omega\Delta$. $M=15$. A-- $\rho=0.0$, B-- $\rho=0.2$;
C-- $\rho=0.4$. SNR=0.1

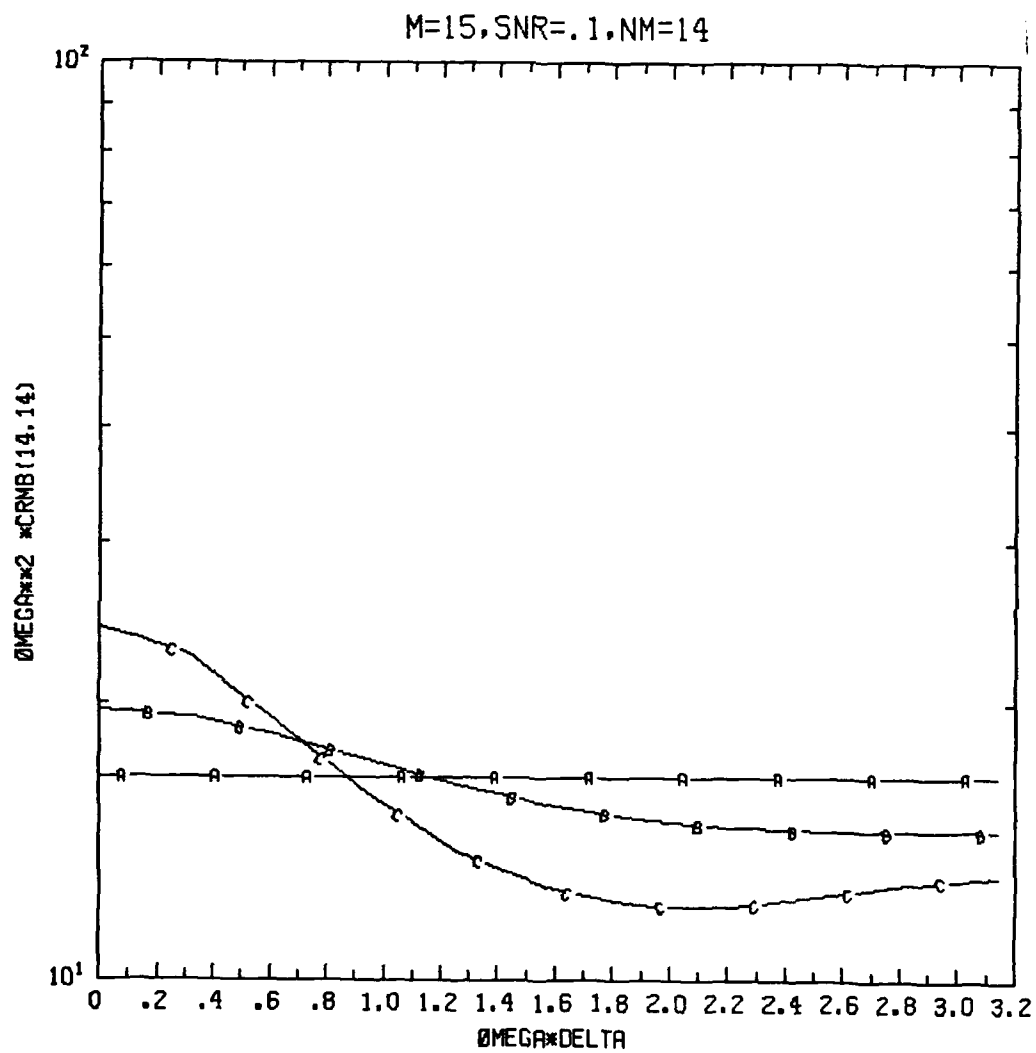


Figure 18. CRMB(14,14) vs $\omega\Delta$. $M=15$. A-- $\rho=0.0$, B-- $\rho=0.2$;
C-- $\rho=0.4$. SNR=0.1

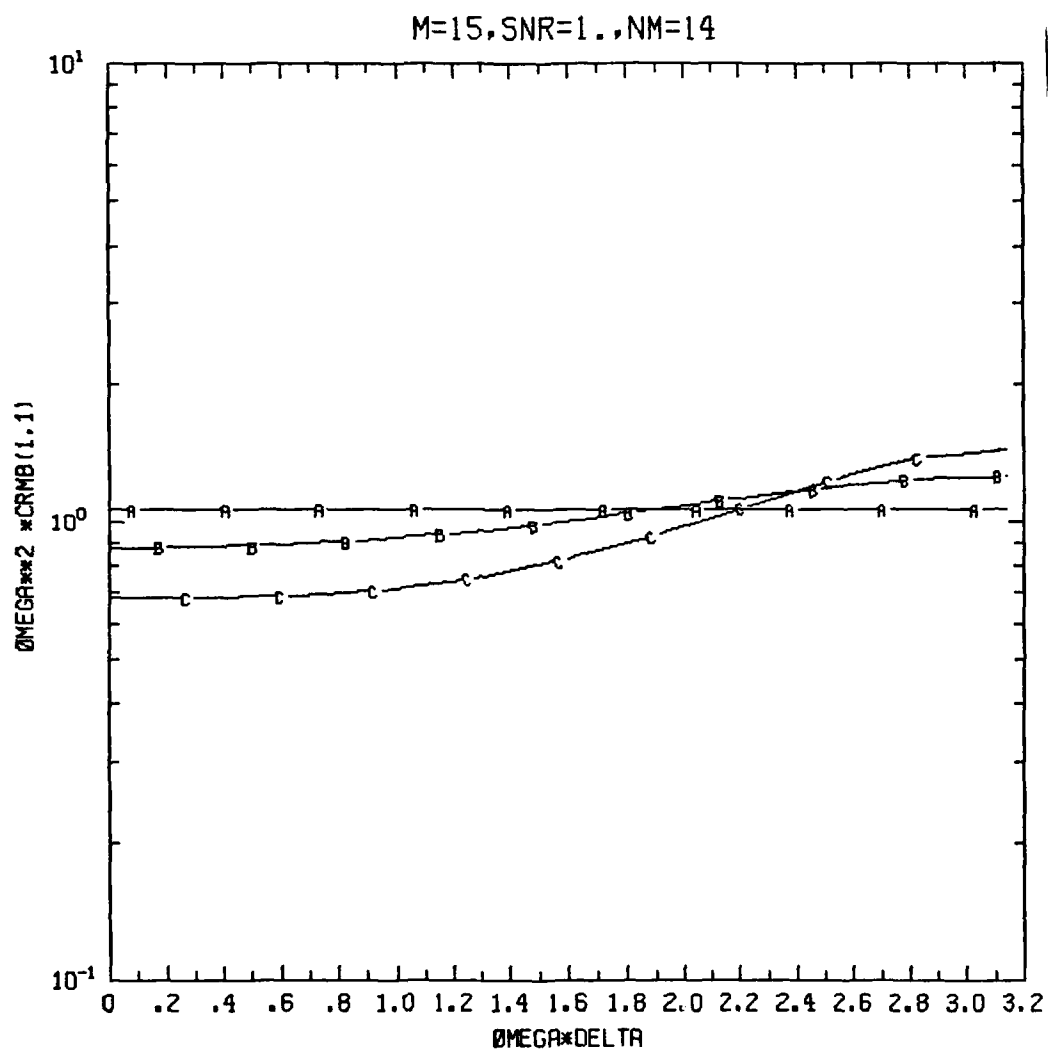


Figure 19. $CRMB(1,1)$ vs $\omega \Delta$. $M=15$. A-- $\rho=0.0$, B-- $\rho=0.2$;
C-- $\rho=0.4$. $SNR=1.0$

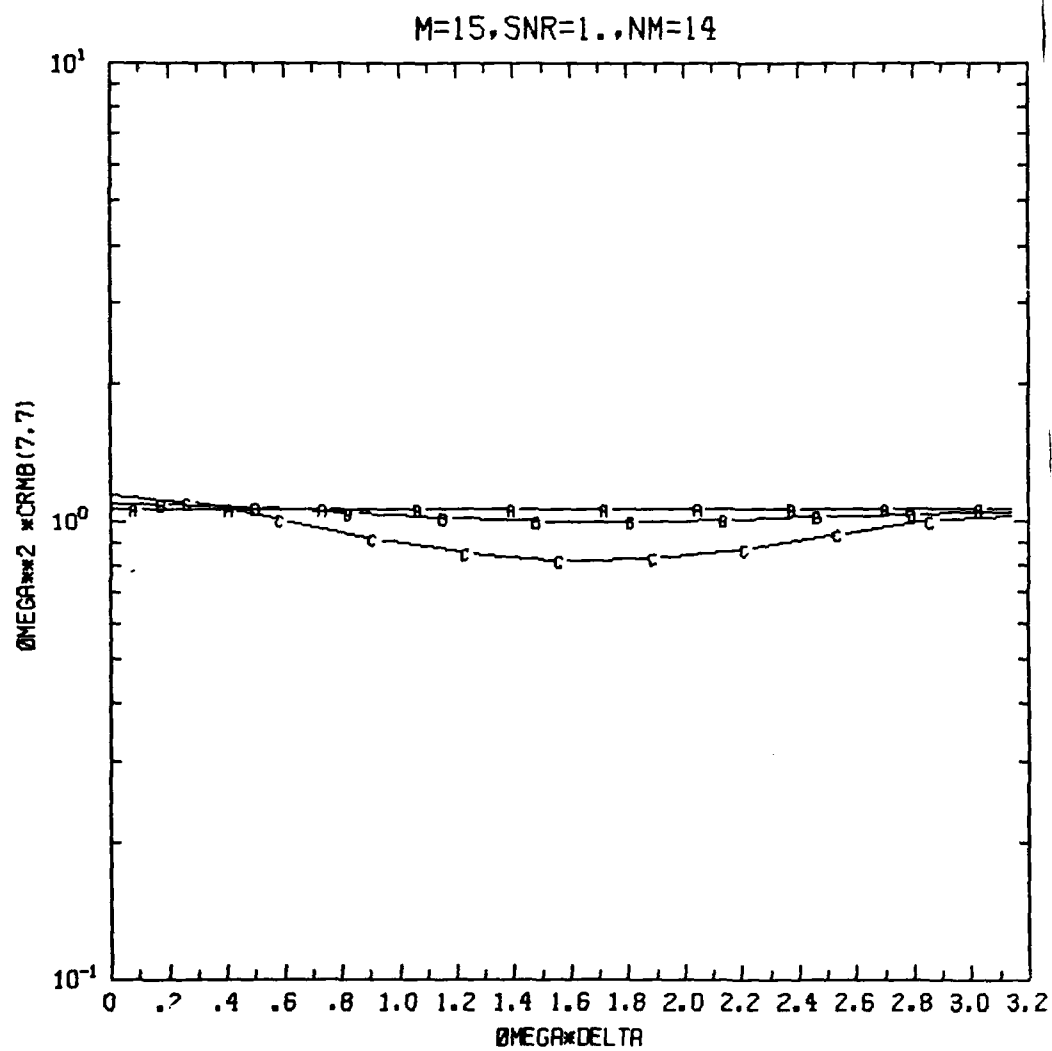


Figure 20. $\text{CRMB}(7,7)$ vs $\omega\Delta$. $M=15$. A-- $\rho=0.0$, B-- $\rho=0.2$;
C-- $\rho=0.4$. $\text{SNR}=1.0$

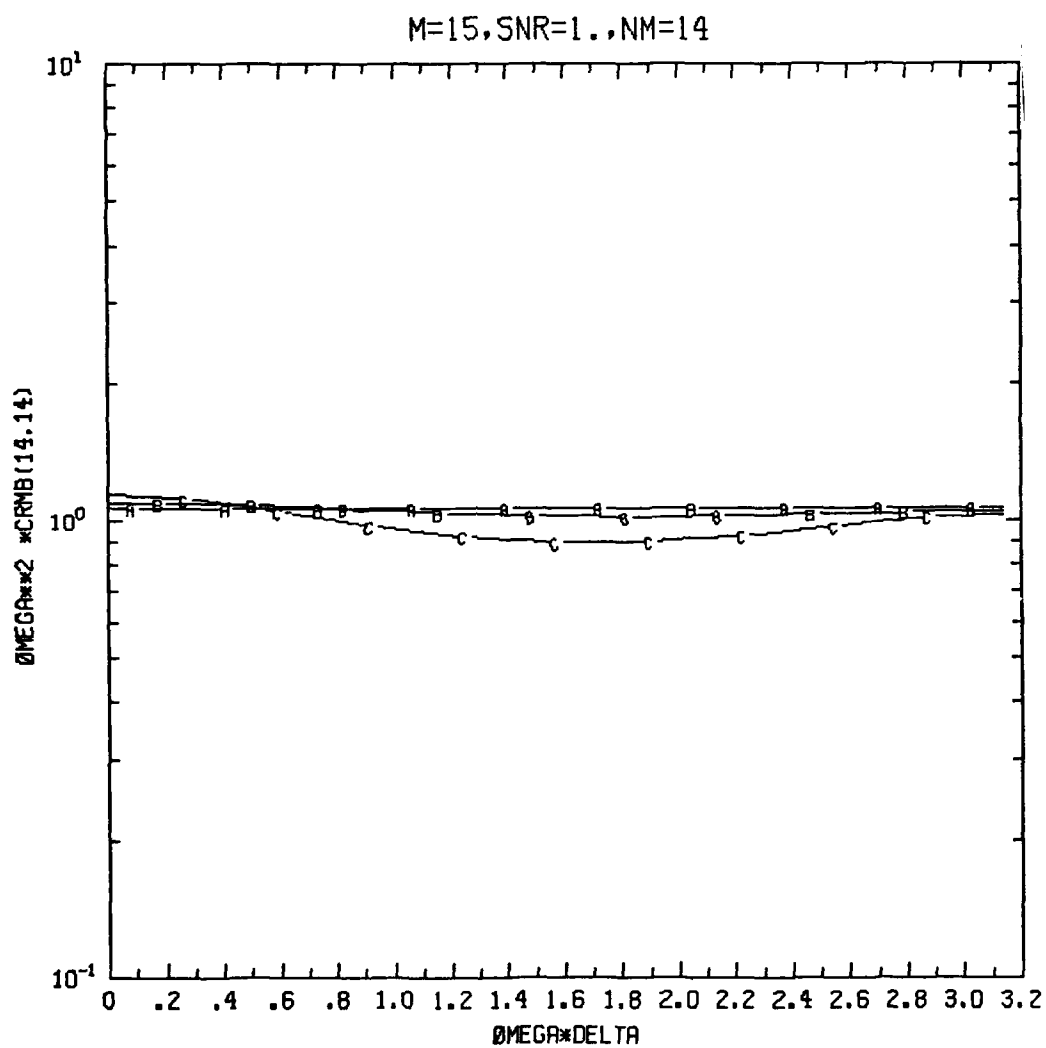


Figure 21. $CRMB(14,14)$ vs $\omega\Delta$. $M=15$. A-- $\rho=0.0$, B-- $\rho=0.2$;
C-- $\rho=0.4$. $SNR=1.0$

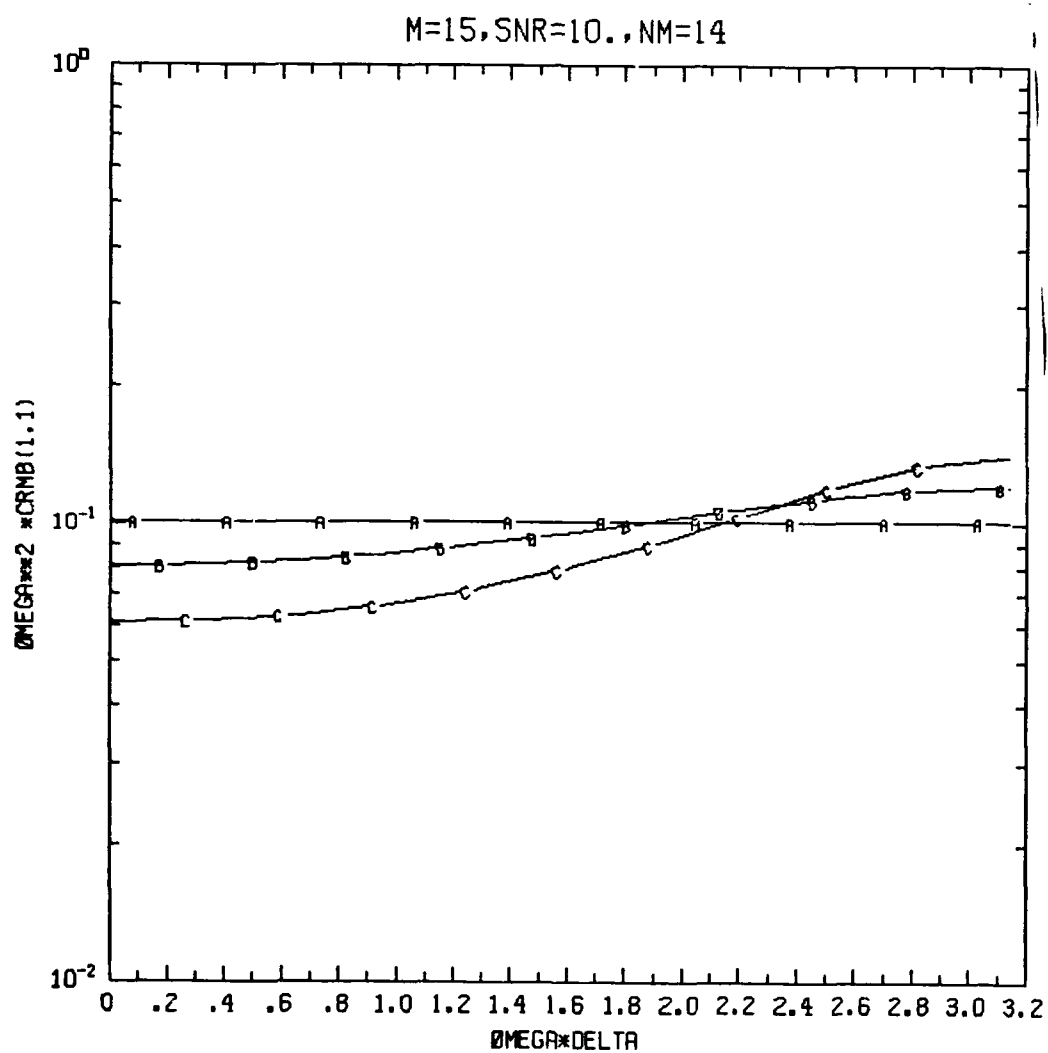


Figure 22. CRMB(1,1) vs $\omega\Delta$. $M=15$. A-- $\rho=0.0$, B-- $\rho=0.2$;
C-- $\rho=0.4$. SNR=10.0

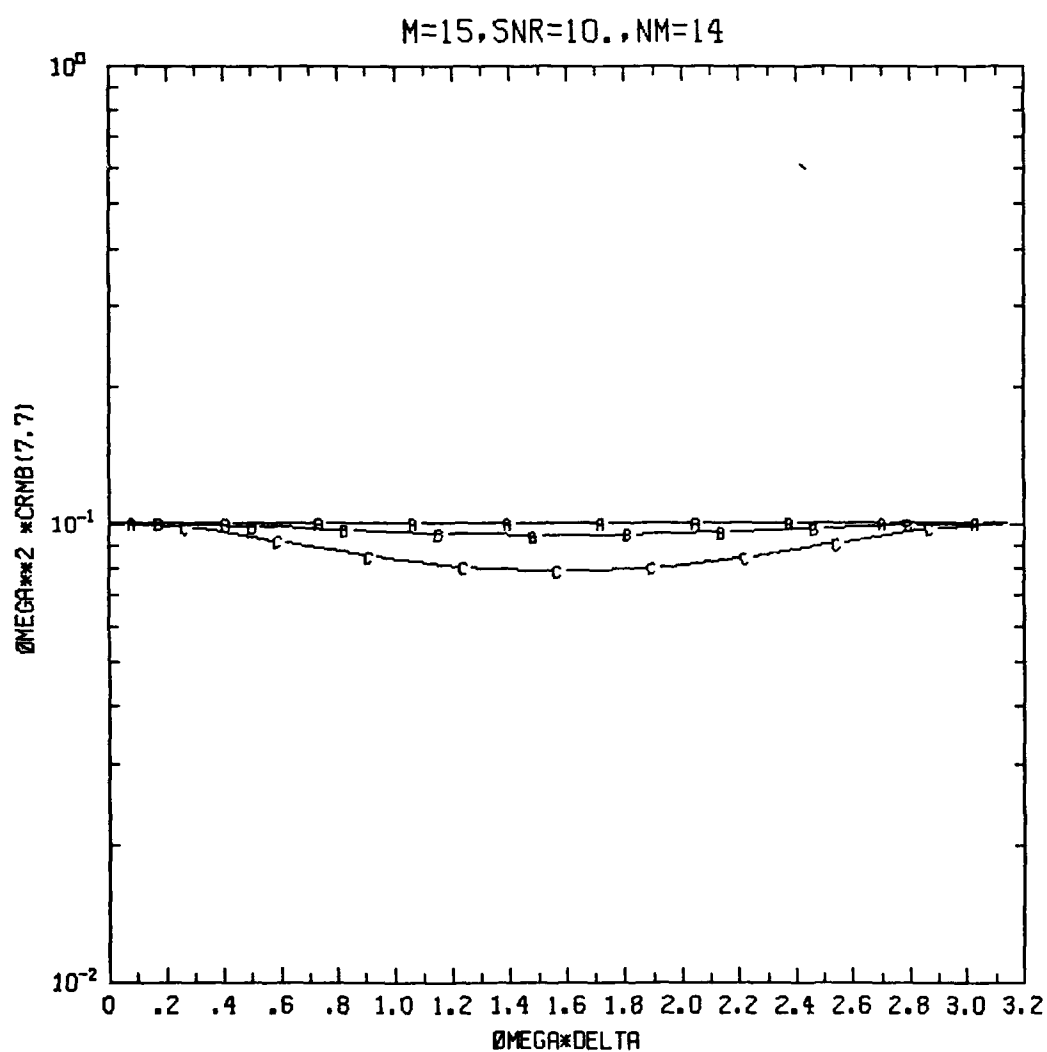


Figure 23. CRMB(7,7) vs $\omega\Delta$. $M=15$. A-- $\rho=0.0$, B-- $\rho=0.2$;
C-- $\rho=0.4$. SNR=10.0

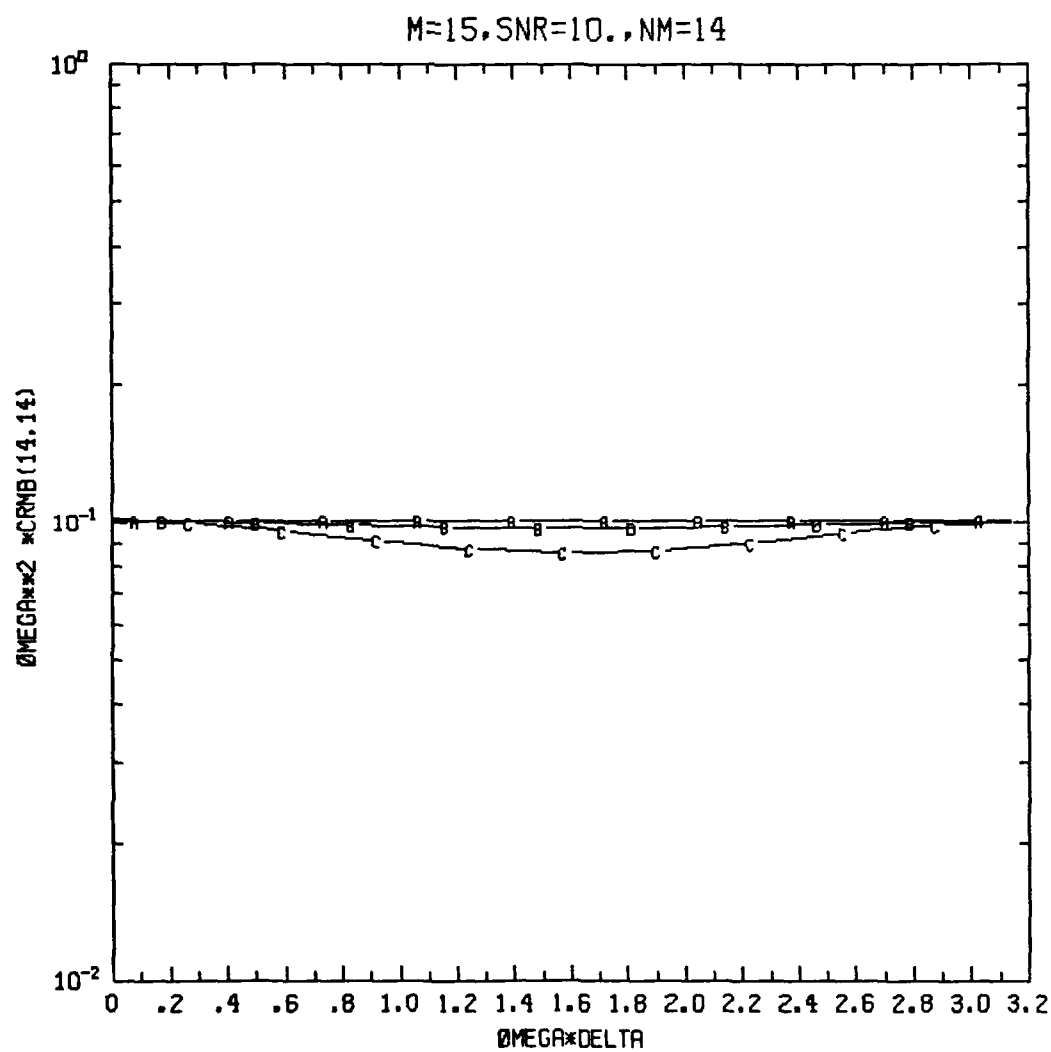


Figure 24. CRMB(14,14) vs $\omega \Delta$. M-15. A-- $\rho=0.0$, B-- $\rho=0.2$;
C-- $\rho=0.4$. SNR=10.0

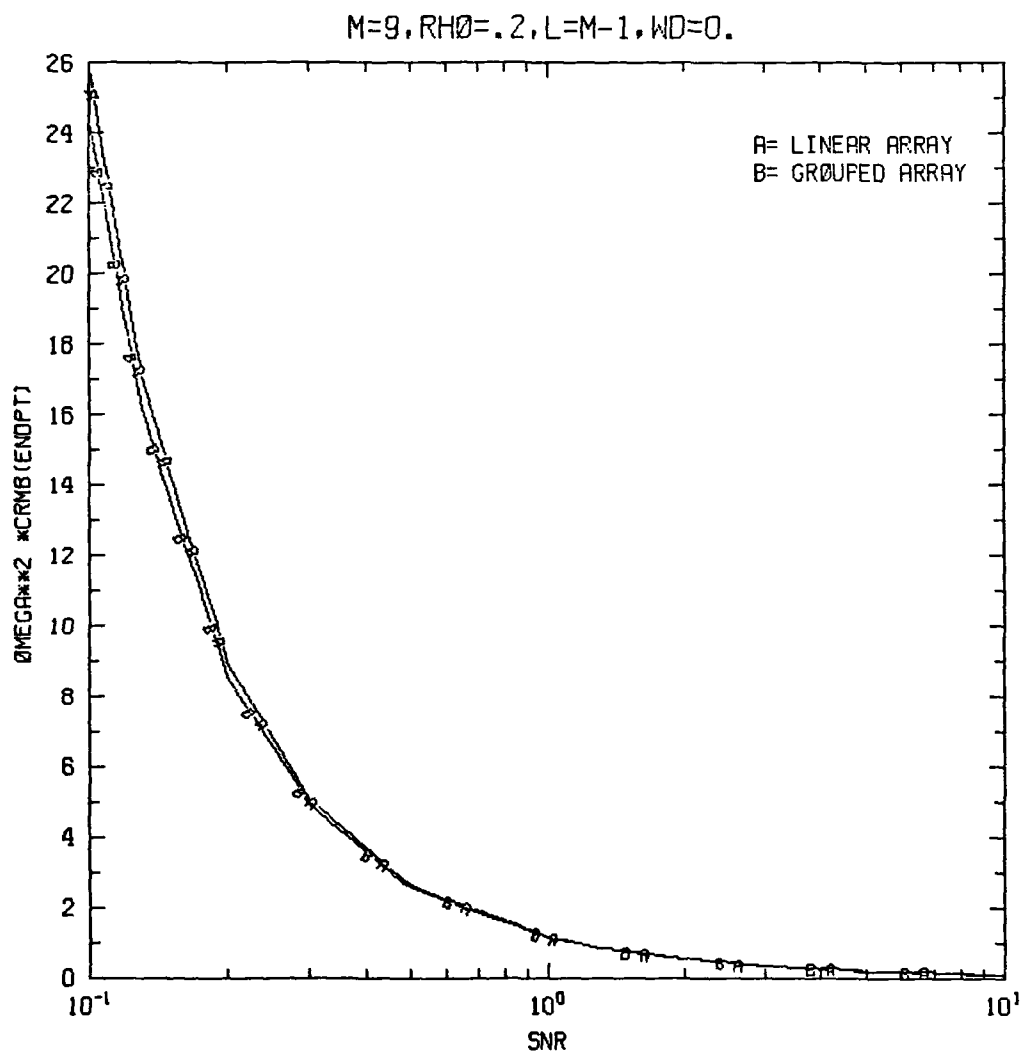


Figure 25. CRMB(8,8) vs SNR. $M=9$, $\rho=0.2$, $\omega\Delta=0$. A -- equally spaced sensors with full 9×9 N-matrix, B--equally spaced sensors with 3×3 block-diagonal N matrix (cluster-independent noise).

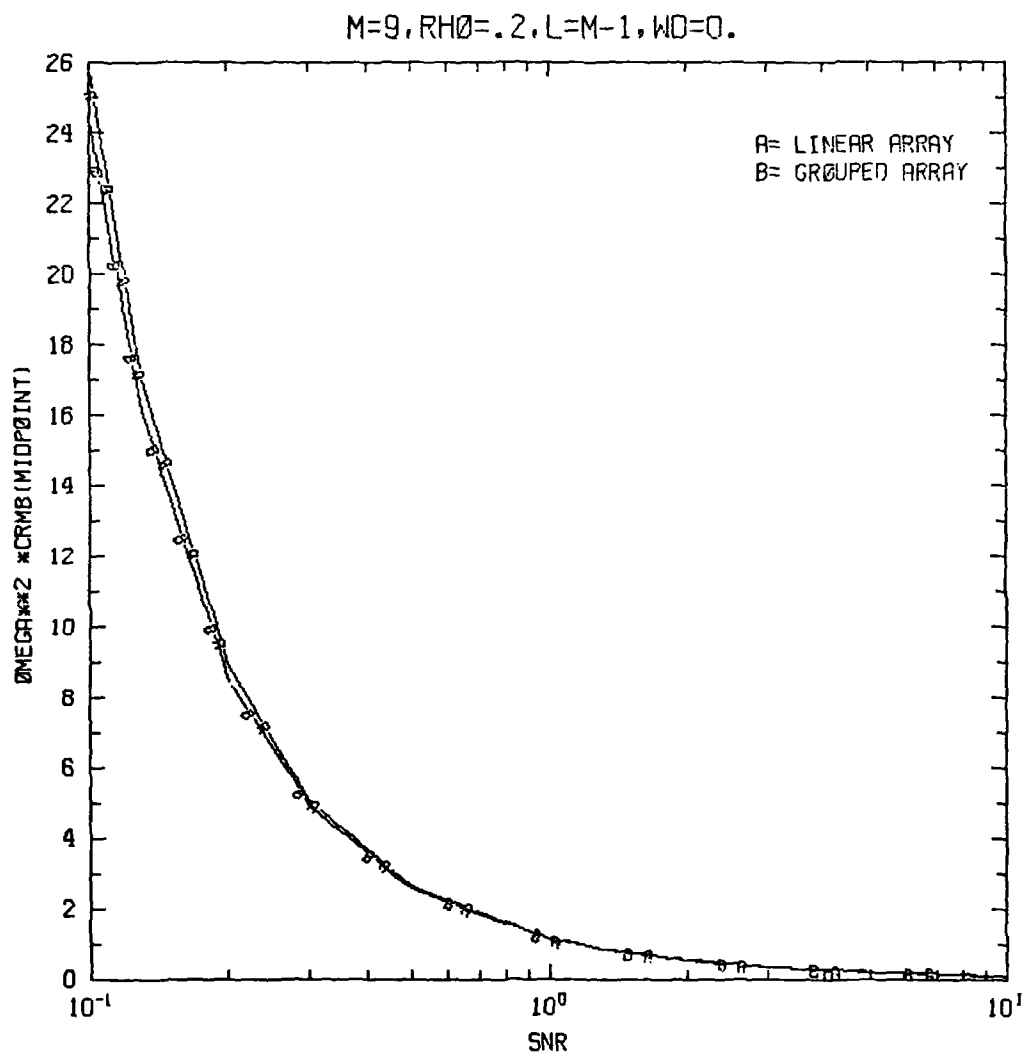


Figure 26. CRMB(4,4) vs SNR. $M=9$, $\rho=0.2$, $\omega=0$. A--equally spaced sensors with full 9×9 N-matrix, B--equally spaced sensors with 3×3 block-diagonal N matrix (cluster-independent noise).

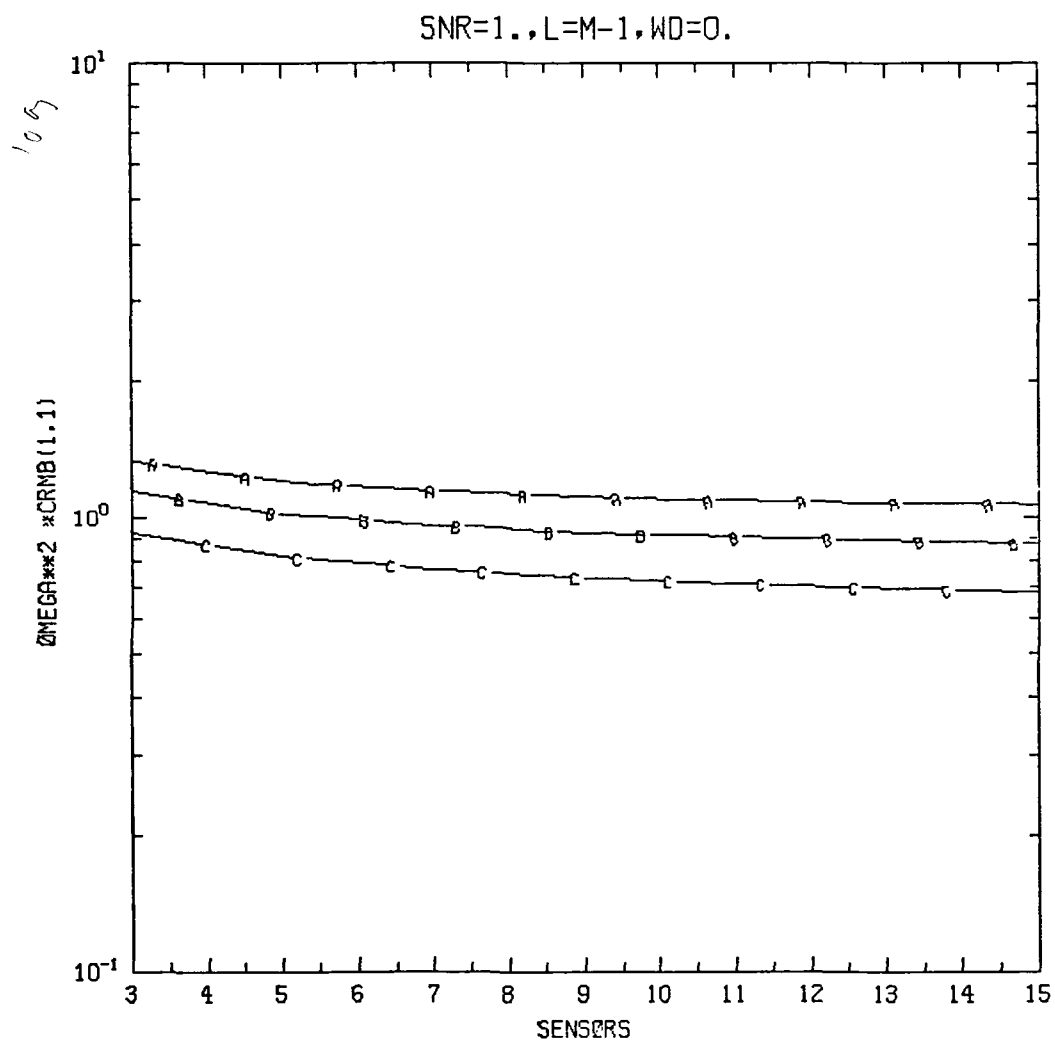


Figure 27. CRMB(1,1) vs M. SNR=1.0. A-- $\rho=0.0$, B-- $\rho=0.2$, C-- $\rho=0.4$. Equally spaced sensors, full N-matrix.

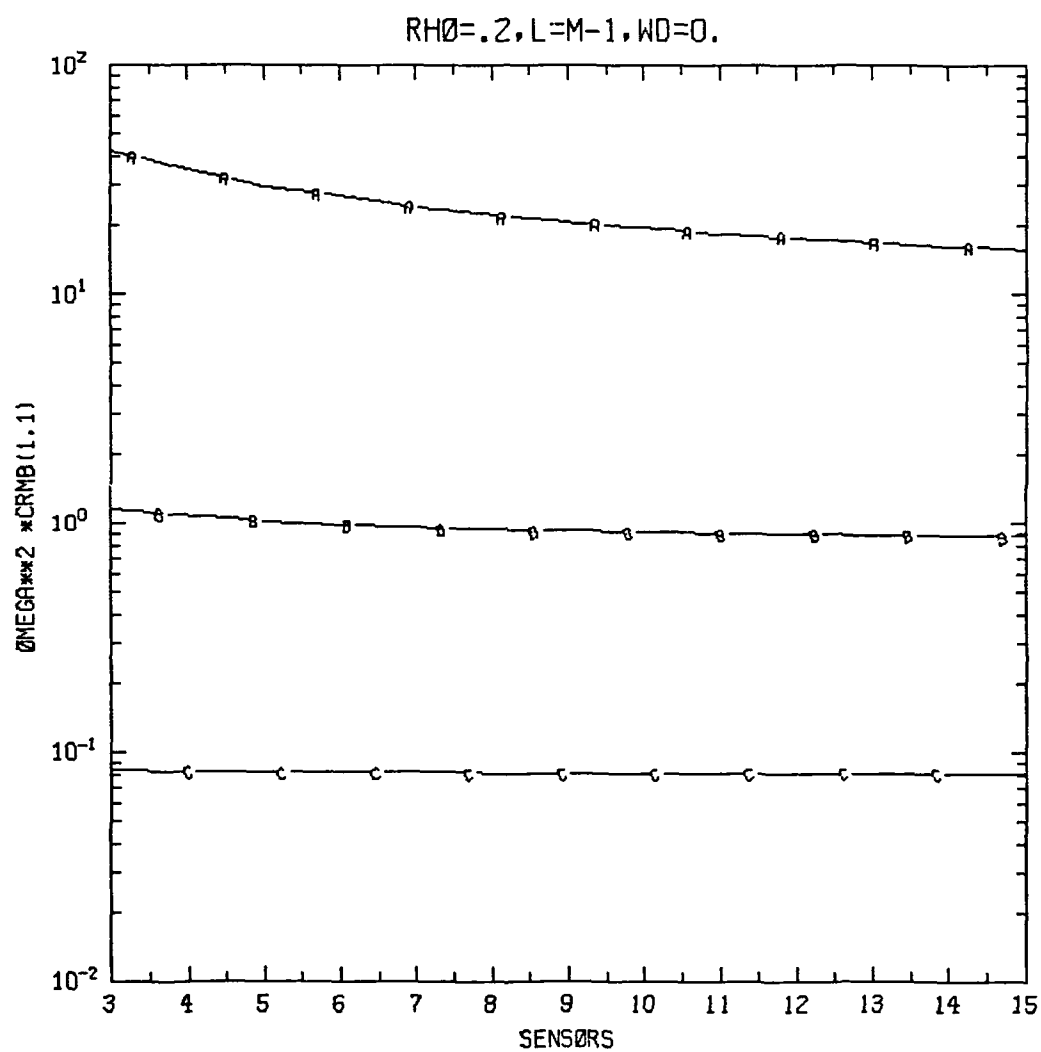


Figure 28. CRMB(1,1) vs M. $\rho=0.2$. A--SNR=0.1, B--SNR=1.0, C--SNR=10.0. Equally spaced sensors, full N matrix.

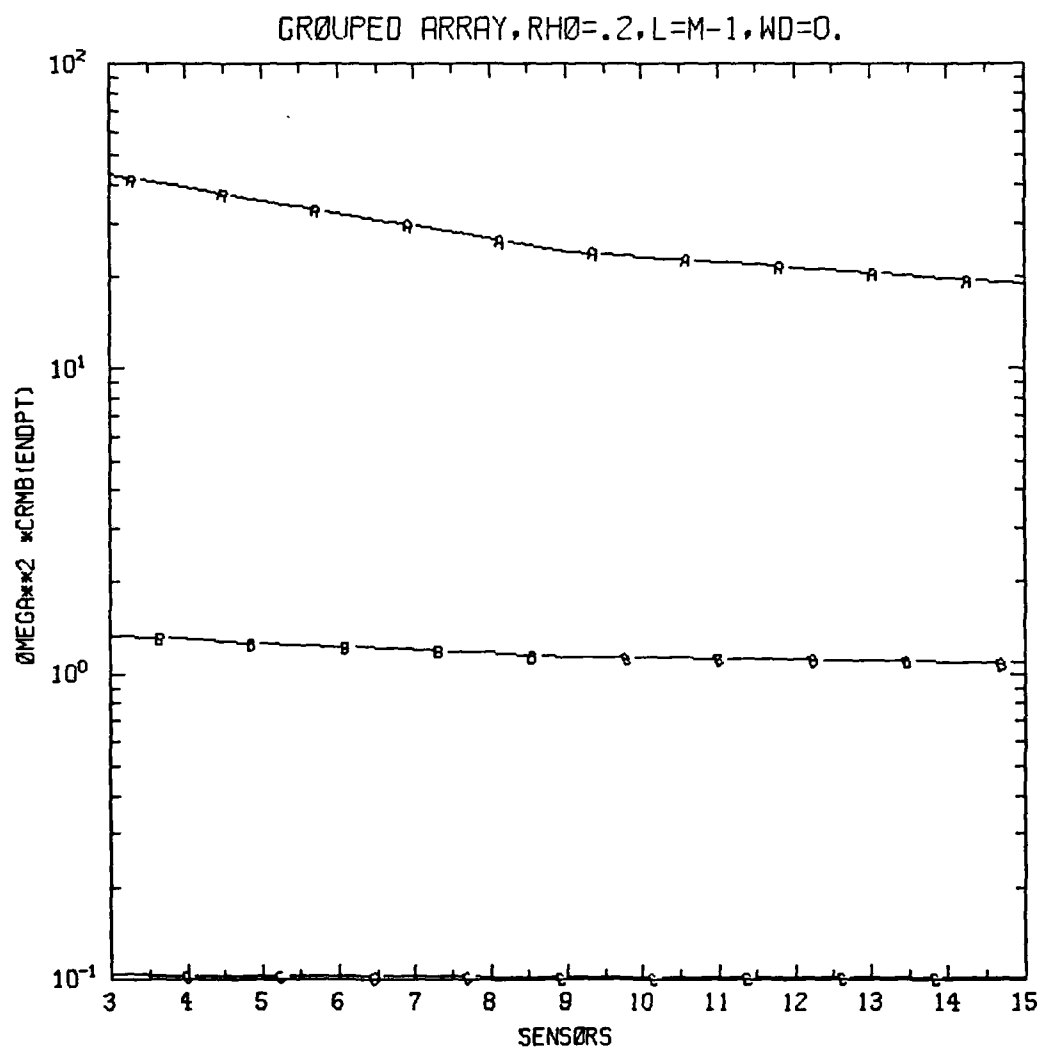


Figure 29. $\text{CRMB}(M-1, M-1)$ vs M . $\rho=0.2$. A--SNR=0.1, B--SNR=1.0, C--SNR=10.0. M sensors have constant array length; clusters of 1, 3, and 5 elements at center and ends of array. No noise correlation between clusters.

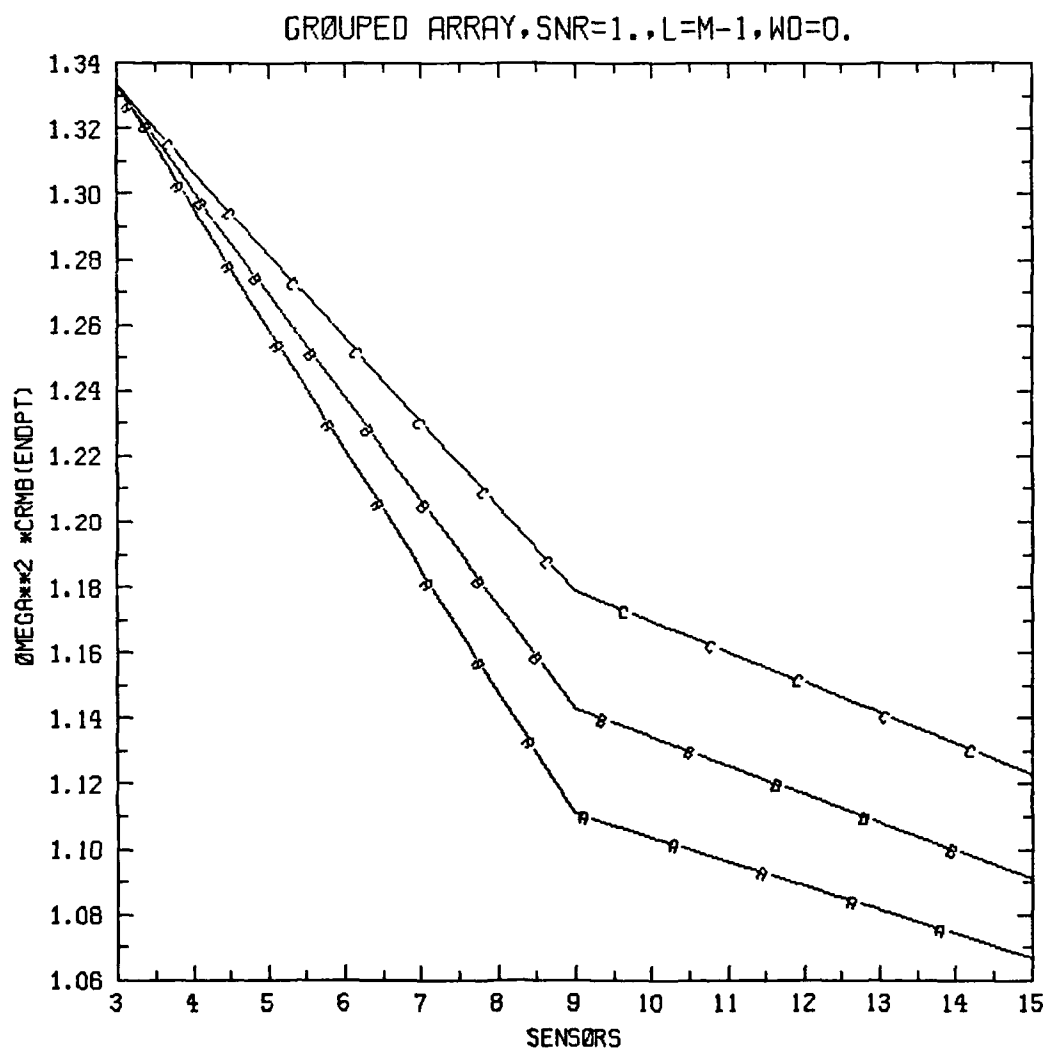


Figure 30. CRMB(M-1, M-1) vs M. SNR=1.0. A-- $\rho=0.0$, B-- $\rho=0.2$, C-- $\rho=0.4$. M sensors have constant array length; clusters of 1, 3, and 5 elements at center and ends of array. No noise correlation between clusters.

VIII. Conclusions

Although a closed-form solution for the ML estimate of the d_{ik} has not been obtained, nonlinear equations which theoretically may be solved are derived. These show considerable complexity which might be somewhat reduced under rather confining conditions. No simple hardware analogy is apparent.

The fact that the ML estimators of the d_{ik} are functions of all the other delays may be a positive observation, that is, no one delay is estimated without consideration of the others. However, it has been shown (at least for uncorrelated noises) by Schultheiss [5] that $M-1$ delays are sufficient in practicality except when all sensors have small SNR. In fact the CRMB may not be found from $(FIM)^{-1}$ using the formulas given herein unless an independent set of d_{ik} is used.

The variances computed for the delays d_i show considerable - several dB - deviation as ρ varies and as M varies. The effect is greater at some look angles than others, and also depends on which delay is considered.

For sufficient spatially correlated noise, clustering sensors is not efficient beyond a certain number. Here we show 6 to 9 sensors is a reasonable number.

Appendix A. Evaluation of the $M(M-1)/2$
Delay Equations for Dispersive Noise

Cron and Sherman [7] have found spatial correlation factors as a function of sensor separation distance and wavelength for diffuse surface noise and for diffuse volume noise. If distances between all sensors considered are small with respect to half-wavelength, the correlations are all essentially a constant. Although long arrays span much more than half a wavelength, a constant correlation is useful when some of the sensors are clustered. At half wavelength the correlation is zero for volume noise, but the zero location varies with other geometrical parameters for surface noise. Both Cron and Sherman [7] and Piersol [8] have suggested exploiting the correlation zero distances to improve delay estimation; the implication being that the less correlation the better. Thus it is reasonable to assume a worst case in which noise at any two sensors has a maximum correlation $\rho = \rho(k)$. Due to the symmetry of the diffuse noise source, there is no time delay associated with the correlation; i.e., all noise cross spectra are real. Thus let the cross spectral matrix for diffuse noise be

$$N = N_1 \begin{bmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ \rho & \dots & 1 \end{bmatrix} \quad (A1)$$

For this N , use of the theorem

$$(I + rc^T)^{-1} = I - \frac{1}{1 + r^T c} rc^T$$

gives

$$N^{-1} = (N^{pq})$$

where

$$\beta = N^{11} = \frac{1}{N_1(1-\rho)} \left(\frac{1 + (M-2)\rho}{1 + (M-1)\rho} \right) \quad (A2a)$$

$$\alpha = N^{pq} = \frac{1}{N_1(1-\rho)} \left(\frac{-\rho}{1 + (M-1)\rho} \right), \quad p \neq q. \quad (A2b)$$

Eq. (19) may be written

$$\sum_{B+} \{ -g_1/g + X^T N^{-1} U N^{-1} X^* \} = 0. \quad (A3)$$

where the elements u_{mn} are given in Eq. (21), and g_1 is in (20c).

A simplification may be obtained by observing the relative magnitude of

$$\begin{aligned} S V^T N^{-1} V^* &= S \sum_p \sum_q N^{pq} e^{j\omega d_{pq}} \\ &= S (M N^{ii} + 2 \sum_p \sum_{q>p} N^{pq} \cos \omega d_{pq}). \end{aligned} \quad (A4)$$

Substituting (A2) into (A4) gives

$$\begin{aligned} S V^T N^{-1} V^* &= \frac{S}{N_1 (1-\rho) (1+(M-1)\rho)} [M(1+(M-2)\rho) \\ &\quad - 2\rho \sum_p \sum_{q>p} \cos \omega d_{pq}] \end{aligned} \quad (A5)$$

Since there are $M(M-1)/2$ terms in the double sum and $|\cos \theta| \leq 1$,

$$\frac{S}{N_1} < \frac{MS/N_1}{1+(M-1)\rho} \leq S V^T N^{-1} V^* \leq \frac{MS/N_1}{(1-\rho)(1+(M-1)\rho)} [1+2((M-3/4)^2 - \frac{\pi}{16})\rho/M] \quad (A6)$$

Thus under the very tolerable conditions that $MS/(N_1(1+(M-1)\rho)) \gg 1$ or

$\rho \ll (\frac{MS}{N_1} - 1)/(M-1)$, we note that in (21)

$$\begin{aligned} \left| \frac{2SN^{ik}\omega \sin \omega d_{ik}}{1+SV^T N^{-1} V^*} \right| &\leq \left| \frac{-2\rho \sin \omega d_{ik}}{N_1(1-\rho)(1+(M-1)\rho)} / (1 + \frac{MS}{N_1(1+(M-1)\rho)}) \right| \\ &\approx \left| \frac{2\rho \sin \omega d_{ik}}{M(1-\rho)} \right| \end{aligned} \quad (A7)$$

Using (A7) in (21) gives

$$u_{mn} \approx \frac{1}{g} \begin{cases} \omega e^{j\omega d_{ik}(j - \frac{2\rho \sin \omega d_{ik}}{m(1-\rho)})}, & (m,n) = (i,k) \\ \omega e^{-j\omega d_{ik}(-j - \frac{2\rho \sin \omega d_{ik}}{m(1-\rho)})}, & (m,n) = (k,i) \\ \frac{-2\rho \omega e^{j\omega d_{mn}} \sin \omega d_{ik}}{M(1-\rho)}, & (m,n) \neq (i,k), (k,i) \end{cases} \quad (A8)$$

The following steps will lead to an equation, nonlinear, for d_{ik} in terms of the other delays and having only real quantities. Thus the d_{ik} terms are separated from others in the summations. The set of $M(M-1)/2$ equations could theoretically be solved for any $M-1$ independent unknowns d_{ik} . Further, assumptions about large M , small ρ , or small d_{ik} lead to simplifications and linearizations, but these are not pursued here.

Using the approximations (A8) in (A3) gives

$$\begin{aligned} \sum_{B+} \frac{1}{g} \{ 2\omega N^{ik} \sin \omega d_{ik} + X^T N^{-1} U N^{-1} X^T \} \\ = \sum_{B+} \frac{1}{g} [2\omega N^{ik} \sin \omega d_{ik} \\ - \frac{2\rho \omega \sin \omega d_{ik}}{M(1-\rho)} \sum_p \sum_q \sum_{(r,t) \neq (k,l), (i,k)} X_p X_q^* N^{pr} N^{tq} e^{j\omega d_{rt}} \\ + \omega \sum_p \sum_q (j - \frac{2\rho \sin \omega d_{ik}}{M(1-\rho)}) X_p X_q^* N^{pi} N^{kq} e^{j\omega d_{ik}} \\ + (-j - \frac{2\rho}{M(1-\rho)}) X_p X_q^* N^{pk} N^{iq} e^{-j\omega d_{ik}}] = 0 \end{aligned} \quad (A9)$$

Grouping index pairs (p,q) according to matches with either m or n or both in $e^{\pm j\omega d_{mn}}$ and using α and β for the N^{-1} elements gives (see Appendix B)

$$\begin{aligned} \sum_{B+} \frac{1}{g} \{ 2\omega N^{ik} \sin \omega d_{ik} \\ - \frac{2\rho \omega \sin \omega d_{ik}}{M(1-\rho)} \sum_{(r,t) \neq (i,k), (k,i)} [(2\alpha\beta(|X_r|^2 + |X_t|^2) \\ + 2\alpha^2 \sum_{p \neq r, t; q \neq r, t} X_p X_q^*) \cos \omega d_{rt} \\ + (\beta^2 X_r X_t^* + \alpha^2 X_t X_r^* + \alpha\beta \sum_q X_r X_q^* + \alpha^2 \sum_q X_t X_q^* \\ + \alpha^2 \sum_{p'} X_p X_{r'}^* + \alpha\beta \sum_{p'} X_p X_{t'}^*) e^{j\omega d_{rt}} \\ + (\alpha^2 X_r X_t^* + \beta^2 X_t X_r^* + \alpha^2 \sum_q X_r X_q^* + \alpha\beta \sum_q X_t X_q^* \\ + \alpha\beta \sum_{p'} X_p X_{r'}^* + \alpha^2 \sum_{p'} X_p X_{t'}^*) e^{-j\omega d_{rt}} \} \end{aligned}$$

$$\begin{aligned}
& - \omega \left(\frac{2\rho \sin \omega d_{ik}}{M(1-\rho)} \right) (2\alpha\beta(|X_i|^2 + |X_k|^2) + 2\alpha^2 \sum_{p'} \sum_{q'} X_p X_q^*) \cos \omega d_{ik} \\
& - \omega (2\alpha\beta(|X_i|^2 + |X_k|^2) + 2\alpha^2 \sum_{p'} \sum_{q'} X_p X_q^*) \sin \omega d_{ik} \\
& + \omega \left(j - \frac{2\rho \sin \omega d_{ik}}{M(1-\rho)} \right) [\beta^2 X_i X_k^* + \alpha^2 X_k X_i^* + \alpha\beta \sum_{q'} X_i X_q^* + \alpha^2 \sum_{q'} X_k X_q^* \\
& \quad + \alpha^2 \sum_{p'} X_p X_i^* + \alpha\beta \sum_{p'} X_p X_k^*] e^{j\omega d_{ik}} \\
& + \omega \left(-j - \frac{2\rho \sin \omega d_{ik}}{M(1-\rho)} \right) [\alpha^2 X_i X_k^* + \beta^2 X_k X_i^* + \alpha^2 \sum_{q'} X_i X_q^* + \alpha\beta \sum_{q'} X_k X_q^* \\
& \quad + \alpha\beta \sum_{p'} X_p X_i^* + \alpha^2 \sum_{p'} X_p X_k^*] e^{-j\omega d_{ik}} = 0 \tag{A10}
\end{aligned}$$

where p' and q' indicate omission in their sequences of r and t for the $e^{j\omega d_{rt}}$ terms or i and k for the $e^{j\omega d_{ik}}$ terms.

Noting that the coefficients of $e^{\pm j\omega d_{mn}}$ are complex conjugates, we may write, using

$$(a + jb)(\cos\theta + j\sin\theta) = a \cos\theta - b \sin\theta + j(b \cos\theta + a \sin\theta)$$

$$(a - jb)(\cos\theta - j\sin\theta) = a \cos\theta - b \sin\theta - j(b \cos\theta + a \sin\theta)$$

$$\sum_{B+} \frac{1}{g} \{ 2\omega \alpha \sin \omega d_{ik}$$

$$\begin{aligned}
& \frac{-2\rho \sin \omega d_{ik}}{M(1-\rho)} \sum_{\substack{(r,t) \neq (i,k), (k,i) \\ t \geq r}} \sum_{t \geq r} \mu [(\alpha\beta(|X_r|^2 + |X_t|^2) \\
& \quad + \alpha^2 \sum_{p'} \sum_{q'} X_p X_q^*) \cos \omega d_{rt} \\
& + ((\alpha^2 + \beta^2) \operatorname{Re}\{X_r X_t^*\} + (\alpha\beta + \alpha^2) \sum_{p'} \operatorname{Re}\{X_r X_p^* + X_t^* X_p\} \cos \omega d_{rt} \\
& - ((\beta^2 - \alpha^2) \operatorname{Im}\{X_r X_t^*\} + (\alpha\beta - \alpha^2) \sum_{p'} \operatorname{Im}\{X_r X_p^* + X_t^* X_p\} \sin \omega d_{rt}] \\
& - 2\omega [(\alpha\beta(|X_i|^2 + |X_k|^2) + \alpha^2 \sum_{p'} \sum_{q'} X_p X_q^*) \sin \omega d_{ik} \\
& + \left(\frac{2\rho \sin \omega d_{ik}}{M(1-\rho)} \right) (\alpha\beta(|X_i|^2 + |X_k|^2) + \alpha^2 \sum_{p'} \sum_{q'} X_p X_q^*) \cos \omega d_{ik}
\end{aligned}$$

$$\begin{aligned}
& + ((\beta^2 - \alpha^2) I_m \{X_i X_k^*\} + (\alpha\beta - \alpha^2) \sum_p I_m \{X_i X_p^* + X_p X_k^*\}) \cos \omega_{ik} \\
& - \left(\frac{2\rho \sin \omega_{ik}}{M(1-\rho)} \right) ((\beta^2 - \alpha^2) I_m \{X_i X_k^*\} + (\alpha\beta - \alpha^2) \sum_p I_m \{X_i X_p^* + X_p X_k^*\}) \sin \omega_{ik} \\
& + ((\alpha^2 + \beta^2) \operatorname{Re} \{X_i X_k^*\} + (\alpha\beta + \alpha^2) \sum_p \operatorname{Re} \{X_i X_p^* + X_p X_k^*\}) \sin \omega_{ik} \\
& + \left(\frac{2\rho \sin \omega_{ik}}{M(1-\rho)} \right) ((\alpha^2 + \beta^2) \operatorname{Re} \{X_i X_k^*\} + (\alpha\beta + \alpha^2) \sum_p \operatorname{Re} \{X_i X_p^* + X_p X_k^*\}) \cos \omega_{ik} \} = 0
\end{aligned} \tag{A11}$$

where $\mu = 1$ if $r = t$, $\mu = 2$ otherwise.

Note that all elements in Eq. (A11) are real because for example

$$\sum_p \sum_q X_p X_q^* = \sum_{p \neq i, k} |X_p|^2 + 2 \sum_{p, q, p' > p'} \operatorname{Re} \{X_p X_q^*\}$$

First simplifications of (A11) may be obtained by observing $\beta \gg |\alpha|$ when $1 + (M-2)\rho \gg |\rho|$, which is often true. The $2\rho \sin \omega_{ik}$ term is negligible with respect to any of the β^2 or $\beta\alpha$ terms, particularly at large SNR. Also

at large SNR, terms such as $\operatorname{Re} \{X_r X_p^*\}$ are approximately equal to $S \cos \omega_{rp}$, and terms such as $I_m \{X_r X_p^*\}$ are approximately $S \sin \omega_{rp}$. If $2\rho/(M(1-\rho)) \ll 1$, many terms drop out.

If the double sum over r, t could be omitted, and if $\omega_{ik} \ll 1$, the equation could be linearized, but this is not generally feasible. There are $M(M-1)/2$ terms in the double sum, each of comparable magnitude to the d_{ik} terms, and the double sum is multiplied by $2\rho \sin \omega_{ik}/(M(1-\rho))$. So roughly to drop out the double sum over r, t , we require $\frac{M(M-1)}{2} \cdot \frac{2\rho \sin \omega_{ik}}{M(1-\rho)} \ll 1$

or $(M-1)\rho/(1-\rho) \ll 1$. This is not likely.

Appendix B. Separation of Terms, Diffuse Noise

In Eq. (A9) the double sum over r, t is broken into parts for $(r, t) \neq (i, k), (k, i)$ and $(r, t) = (i, k), (k, i)$. Each part then has terms such as

$$e^{j\omega d_{ik}} \sum_p \sum_q X_p X_q^* N^{pi}_{ik} N^{kq} + e^{-j\omega d_{ik}} \sum_p \sum_q X_p X_q^* N^{pk}_{ik} N^{iq}$$

wherein r and t replace i and k for the other part of (A9). The double sum over p, q may further be broken down into parts for which p or q or both are equal to i or k or both, each case giving different values for N^{pi} or N^{kq} . These give the following results, using β and α from (A2) and e^+ and e^- for the exponentials. (Similar results are obtained for the difference of the conjugate exponentials, yielding $\sin \omega d_{ik}$ terms).

$$\underline{p, q = i, i}$$

$$\beta \alpha |X_i|^2 e^+ + \beta \alpha |X_i|^2 e^- = 2\beta \alpha |X_i|^2 \cos \omega d_{ik}$$

$$\underline{p, q = k, k}$$

$$2\beta \alpha |X_k|^2 \cos \omega d_{ik}$$

$$\underline{p, q = i, k}$$

$$\beta^2 X_i X_k^* e^+ + \alpha^2 X_i X_k^* e^-$$

$$\underline{p, q = k, i}$$

$$\alpha^2 X_k X_i^* e^+ + \beta^2 X_k X_i^* e^-$$

$$\underline{p, q = i, q \neq i, k}$$

$$\sum_q X_i X_q^* \beta \alpha e^+ + \sum_q X_i X_q^* \alpha^2 e^-$$

$$\underline{p, q = k, q \neq i, k}$$

$$\sum_q X_k X_q^* \alpha^2 e^+ + \sum_q X_k X_q^* \beta \alpha e^-$$

$$\underline{p, q = p \neq i, k; q = i}$$

$$\sum_p X_p X_i^* \alpha^2 e^+ + \sum_p X_p X_i^* \alpha \beta e^-$$

$$\underline{p, q = p \neq i, k; q = k}$$

$$\sum_p x_p x_k^* \alpha \beta e^+ + \sum_p x_p x_k^* \alpha^2 e^-$$

$$\underline{p \neq i, k; q \neq i, k}$$

$$2 \sum_{p \neq i, k} \sum_{q \neq i, k} \alpha^2 x_p x_q^* \cos \omega d_{ik}$$

Appendix C. Evaluation of FIM for Diffuse Noise,
An Independent Subset of $M(M-1)/2$ Delays

The following applies to any subset of $M-1$ independent delays d_{ik} .

To determine the FIM, we must calculate the expected values of $X_r X_t^*$ using the noise matrix in (A1). This gives

$$\langle X_r X_t^* \rangle = \begin{cases} S + N_1, & r = t \\ S e^{-j\omega d_{rt}} + \rho N_1, & r \neq t \end{cases} \quad (C1)$$

We also note that (23) becomes for real N

$$-\frac{\partial^2 \Lambda''}{\partial d_{ik}^2} = - \sum_{B+} \{ 2\omega^2 g N^{ik} \cos \omega d_{ik} + 4\omega^2 (N^{ik})^2 \sin^2 \omega d_{ik} \} / g^2 + X^T N^{-1} A(m,n) N^{-1} X^* \quad (C2)$$

The elements of $A(m,n) = (a_{mn})$ are (using (24)-(28) and real N)

$$a_{mn} = \left(\frac{1}{g} \left(\frac{\partial^2 V^* V^T}{\partial d_{ik}^2} \right)_{m,n} + \frac{2\omega^2 N^{ik} \cos \omega d_{ik}}{g^2} (V^* V^T)_{m,n} + \frac{4\omega^2 N^{ik} \sin \omega d_{ik}}{g^2} \left(\frac{\partial V^* V^T}{\partial d_{ik}} \right)_{m,n} + \frac{8\omega^2 (N^{ik})^2 \sin^2 \omega d_{ik}}{g^3} (V^* V^T)_{m,n} \right)$$

$$= \frac{1}{g} \left\{ \begin{aligned} & -\omega^2 e^{j\omega d_{ik}} + \frac{2\omega^2 N^{ik} \cos \omega d_{ik} e^{j\omega d_{ik}}}{g} \\ & + j \frac{4\omega^2 N^{ik} \sin \omega d_{ik} e^{j\omega d_{ik}}}{g} \\ & + \frac{8\omega^2 (N^{ik})^2 \sin^2 \omega d_{ik} e^{j\omega d_{ik}}}{g^2}, \quad (m,n) = (i,k) \\ & \frac{2\omega^2 N^{ik} \cos \omega d_{ik} e^{j\omega d_{mn}}}{g} + \frac{8\omega^2 (N^{ik})^2 \sin^2 \omega d_{ik} e^{j\omega d_{mn}}}{g^2}, \quad (m,n) \neq (i,k), (k,i) \end{aligned} \right. \quad (C3)$$

and $a_{nm} = a_{mn}^*$.

Now from (C2) we may expand

$$\begin{aligned} \langle X^T N^{-1} A N^{-1} X^* \rangle &= \sum_m \sum_n \langle \sum_r X_r N^{rm} \sum_t a_{mn} N^{nt} X_t^* \rangle \\ &= \sum_m \sum_n \sum_r \sum_t \langle X_r X_t^* \rangle N^{rm} N^{nt} a_{mn} \end{aligned} \quad (C4)$$

Applying (C1), (C3) and (C4) to (C2) and using the separation-of-terms process in Appendix B gives for the diagonal elements of the FIM,

$$\begin{aligned} (\text{FIM})_{vv} &= - \sum_{B+g} \frac{\omega^2}{g} \left\{ 2\alpha \cos \omega d_{ik} + \frac{4\alpha^2}{g} \sin^2 \omega d_{ik} \right. \\ &+ (-1 + \frac{2\alpha \cos \omega d_{ik}}{g} + \frac{8\alpha^2 \sin^2 \omega d_{ik}}{g^2}) [4\alpha\beta(S+N_1) \cos \omega d_{ik} \\ &+ 2(\alpha^2 + \beta^2) \rho N_1 \cos \omega d_{ik} + 2\beta^2 S + 2\alpha^2 S \cos 2\omega d_{ik} \\ &+ 4(M-2) \rho N_1 (\alpha^2 + \alpha\beta) \cos \omega d_{ik} \\ &+ 2S \sum_{q'} (\alpha\beta \cos \omega(d_{ik} - d_{iq'}) + \alpha^2 \cos \omega(d_{iq'} + d_{ik}) + \alpha^2 \cos \omega(d_{ik} - d_{kq'}) \\ &\quad \left. + \alpha\beta \cos \omega(d_{ik} + d_{kq'})) \right. \\ &+ 2\alpha^2((M-2)(S+N_1) + 2((M-2)^2 - (M-2))\rho N_1 \\ &\quad \left. + 2 \sum_{p'} \sum_{q' > p'} S \cos \omega d_{pq'}) \cos \omega d_{ik} \right] \\ &- \left(\frac{4\alpha \sin \omega d_{ik}}{g} \right) [4\alpha\beta(S+N_1) \sin \omega d_{ik} \\ &+ 2(\alpha^2 + \beta^2) \rho N_1 \sin \omega d_{ik} + 2\alpha^2 S \sin 2\omega d_{ik} \\ &+ 4(M-2) \rho N_1 (\alpha^2 + \alpha\beta) \sin \omega d_{ik} \\ &+ 2S \sum_{q'} (\alpha\beta \cos \omega(d_{ik} - d_{iq'}) - \alpha^2 \cos \omega(d_{iq'} + d_{ik}) + \alpha^2 \cos \omega(d_{ik} - d_{kq'}) \\ &\quad \left. - \alpha\beta \cos \omega(d_{ik} + d_{kq'})) \right. \\ &+ 2\alpha^2(M-2)(S+N_1) + 2((M-2)^2 - (M-2)) \rho N_1 + 2 \sum_{p'} \sum_{q' > p'} \cos \omega d_{pq'}) \sin \omega d_{ik} \\ &+ \left(\frac{2\omega^2 \alpha \cos \omega d_{ik}}{g} + \frac{8\omega^2 \alpha^2 \sin^2 \omega d_{ik}}{g^2} \right) \sum_{(r,t) \neq (i,k)} \sum_{(k,i)} [4\alpha\beta(S+N_1) \cos \omega d_{rt} \end{aligned}$$

$$\begin{aligned}
& + 2(\alpha^2 + \beta^2) \rho N_1 \cos \omega d_{rt} + 2\beta^2 S + 2\alpha^2 S \cos 2\omega d_{rt} \\
& + 4(M-2) \rho N_1 (\alpha^2 + \alpha\beta) \cos \omega d_{rt} \\
& + 2 S \sum_{q'} (\alpha\beta \cos \omega(d_{rt} - d_{rq}) + \alpha^2 \cos \omega(d_{rq} + d_{rt}) \\
& \quad + \alpha^2 \cos \omega(d_{rt} - d_{tq}) + \alpha\beta \cos \omega(d_{rt} + d_{rq})) \\
& + 2\alpha^2 ((M-2)(S+N_1) + 2((M-2)^2 - (M-2)) \rho N_1 \\
& \quad + 2 \sum_{p'} \sum_{q' > p'} S \cos \omega d_{pq}) \cos \omega d_{rt}] \} \tag{C5}
\end{aligned}$$

Some reduction in the number of terms may be obtained by gathering coefficients of sines and cosines, but this will not be done here. Rather the various terms are left for better inspection and identification with their sources. However, let it be observed that the diagonal elements $(FIM)_{vv}$ may be written

$$\begin{aligned}
(FIM)_{vv} = & - \sum_{B+} \frac{\omega^2}{g} \{ 2\alpha \cos \omega d_{ik} + \frac{4\alpha^2}{g} \sin^2 \omega d_{ik} \\
& + (-1 + \frac{2\alpha \cos \omega d_{ik}}{g} + \frac{8\alpha^2 \sin^2 \omega d_{ik}}{g^2}) A_1(i,k) \\
& - (\frac{4\alpha \sin \omega d_{ik}}{g}) A_2(i,k) \\
& + (\frac{2\omega^2 \alpha \cos \omega d_{ik}}{g} + \frac{8\omega^2 \alpha^2 \sin^2 \omega d_{ik}}{g^2}) \sum_{(r,t) \neq (i,k), (k,i)} A_3(r,t) \} \tag{C6}
\end{aligned}$$

where A_1 , A_2 , and A_3 are as indicated in (C5). These will be used again in the off-diagonal elements' expression.

The off-diagonals are found similarly.

Utilizing functions A_1 , A_2 , and A_3 as in (C5) and (C4) the off-diagonals are

$$\begin{aligned}
(FIM)_{vw} = & - \sum_{B+} \frac{\omega^2}{g} \{ 4 \alpha^2 \sin \omega d_{ik} \sin \omega d_{rt} / g \\
& + \frac{8\alpha^2 \sin \omega d_{rt} \sin \omega d_{ik}}{g^2} A_1(i,k) \\
& + \frac{2 \alpha \sin \omega d_{rt}}{g} A_2(i,k)
\end{aligned}$$

$$\begin{aligned}
& + \frac{8\alpha^2 \sin\omega d_{rt} \sin\omega d_{ik}}{g^2} A_1(r,t) \\
& + \frac{2\alpha \sin\omega d_{ik}}{g} A_2(r,t) \\
& + \frac{8\alpha^2 \sin\omega d_{rt} \sin\omega d_{ik}}{g^2} \sum_{(m,n) \neq (i,k), (k,i)} \sum_{(r,t), (t,r)} A_3(m,n) \} \quad (C7)
\end{aligned}$$

A few remarks are in order at this point. The obvious feature is that expressions (C6) and (C7) for the FIM elements are very complex; they are functions not only of the delays whose covariance is sought but also of all other delays. One simplification is to discard α^2 and/or α -factor terms with respect to β^2 terms. Another is to consider the case where all delays are equal to zero. For a zero delay vector, the covariances in (C7) are also zero, the inversion of the FIM is considerably simplified, and the diagonal elements of FIM and their inverses are functions of ρ, M, S and N_1 only.

It is interesting that in this zero-delay situation the covariance of two delays is zero. This differs from the ML estimation of the M-1 delays referred to a single sensor (see Eq. 7). Those delays have a non-zero covariance with or without correlated noise. Mathematically the difference is between $\frac{\partial}{\partial d_{rt}} \left(\frac{\partial V^* V^T}{\partial d_{ik}} \right)$ and $\frac{\partial}{\partial d_2} \left(\frac{\partial V^* V^T}{\partial d_1} \right)$.

The second-order partial with double subscripted variables is zero, while that with single subscripts is not. This is because the elements of $V^* V^T$ are of the form $\exp(j\omega(d_i - d_k)) = \exp(j\omega d_{ik})$. For example if $d_1 = d_i$ and

$d_2 = d_k$, then $\frac{\partial}{\partial d_2} \left(\frac{\partial V^* V^T}{\partial d_1} \right)$ yields an element $\omega^2 \exp(j\omega(d_1 - d_2)) \neq 0$, but

$\frac{\partial^2}{\partial d_{rt} \partial d_{ik}} (V^* V^T) = 0$. Further, if $\rho = 0$, then the off-diagonal elements of

the FIM of the d_{ik} are always zero. Evidently with spatially uncorrelated noise, ML estimates of the d_{ik} are uncorrelated, although either the generalized cross-correlator measurements of d_{ik} which yield ML estimates of the d_j [3,4] or the ML estimates of the M-1 delays d_i are correlated, as shown in Eqs. (13) and (7).

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